

Sec. 4: Turing Machines

- (a) Definition of the Turing Machine.
- (b) URM computable functions are Turing computable.
- (c) Undecidability of the Turing Halting Problem

(a) Definition of TMs

- There are two problems with the model of a URM:
- Execution of a single URM instruction might take arbitrarily long:
 - Consider $\text{succ}(n)$.
 - If R_n contains in binary $\underbrace{111 \dots 111}_{k \text{ times}}$, this instruction replaces it by $1 \underbrace{000 \dots 000}_{k \text{ times}}$.
 - We have to replace k symbols 1 by 0.
 - k is arbitrary
→ this single step might take arbitrarily long time.

First Problem of URMs

- That incrementing a number by one takes arbitrarily many steps happens on a real computer as well:
 - If we want to represent arbitrary big numbers on the computer, we have to represent them by multiple machine integers
 - Then incrementing a number by one will correspond to arbitrarily many machine instructions (although usually only a few).
 - However, often in complexity theory this problem is ignored because the effect is marginal in real applications.
 - The exception are applications in which very big integers occur, e.g. tests for primality. There this effect cannot be ignored any more.

First Problem of URMs

- If one takes this effect into account, one needs in many examples to multiply the running time by a factor of $\ln(n)$, where n is the largest number occurring.
- Therefore URMs unsuitable as a basis for defining the precise complexity of algorithms.
- However, there are theorems linking complexity of URMs to actual complexities of algorithms.

Second Problem of URMs

- We aim at a notion of computability, which covers all possible ways of computing something, independently of any concrete machine.
- URMs are a model of computation which covers current standard computers.
- However, there might be completely different notions of computability, based on symbolic manipulations of a sequence of characters, where it might be more complicated to see directly that all such computations can be simulated by a URM.
- It is more easy to see that such notions are covered by the Turing machine model of computation.

Idea of a Turing Machine

- Idea of a Turing machine (TM):
Analysis of a computation carried out by a human being (agent) on a piece of paper.

$$\begin{array}{r} 15 \cdot 16 = \\ 15 \\ \underline{90} \\ 240 \end{array}$$

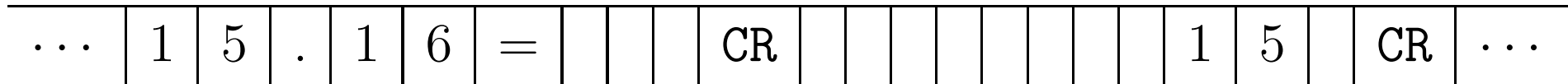
Idea of a Turing Machine

- Steps in this formulation:
 - Algorithm should be deterministic.
 - The agent will use only finitely many symbols, put at discrete positions on the paper.

		1	5	.	1	6	=				
									1	5	
										9	0
									—	—	—
									2	4	0

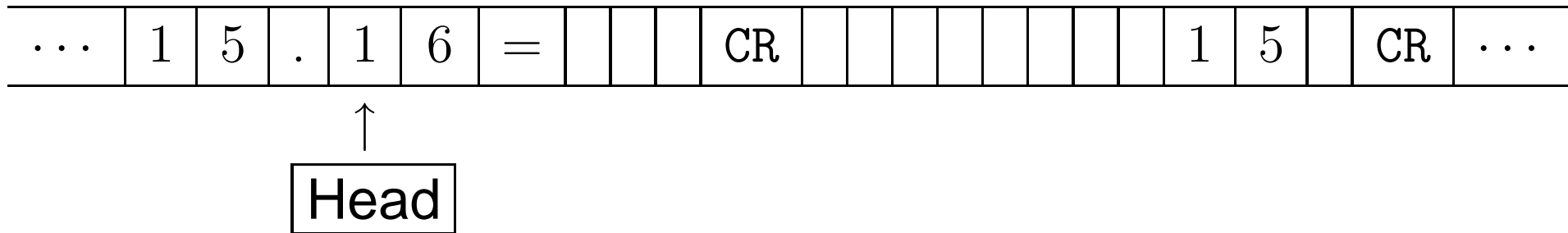
Idea of a Turing Machine

- We can replace a two-dimensional piece of paper by one potentially infinite tape, by using a special symbol for a line break.
- Each entry on this tape is called a cell:



Steps in Formalising TMs

- In the real situation, agent can look at several cells at the same time, but bounded by his physical capability. Can be simulated by looking at one cell only at any time, and moving around in order to get information about neighbouring cells.

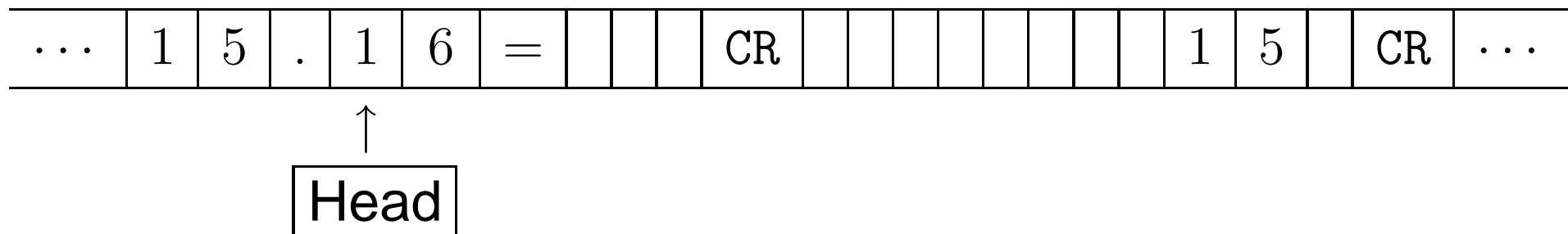


Steps in Formalising TMs

- In the real situation, an agent can make arbitrary jumps, but bounded by the physical ability of the agent.
Each such jump can be simulated by finitely many one-step jumps.
→ Restriction to one-step movements.

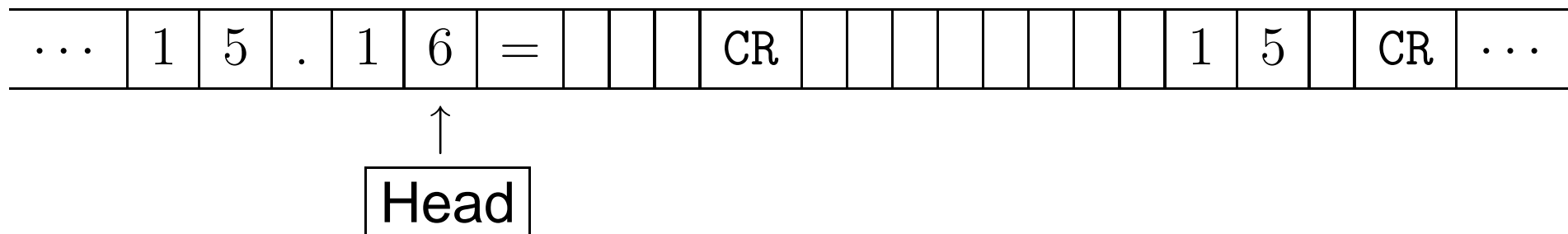
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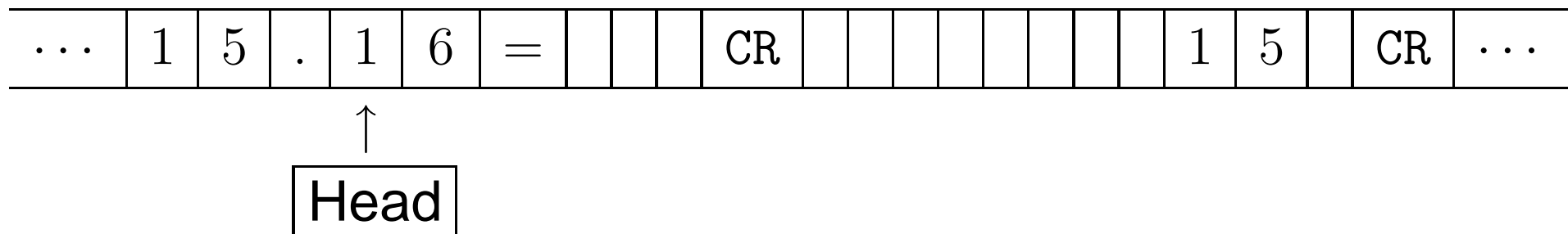
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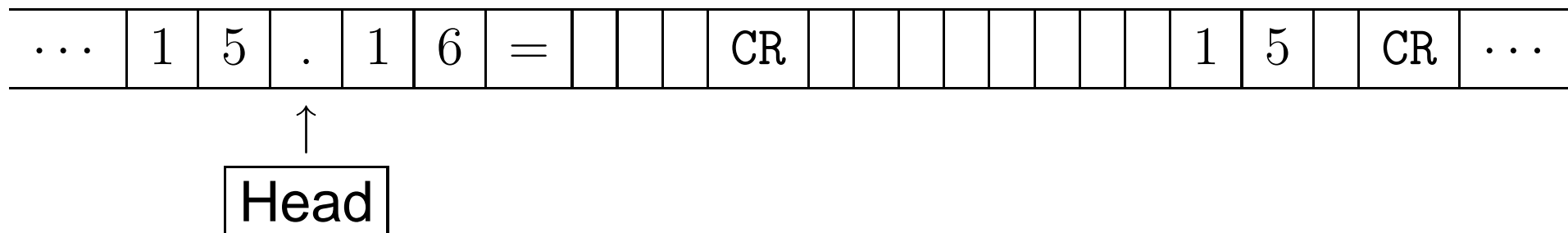
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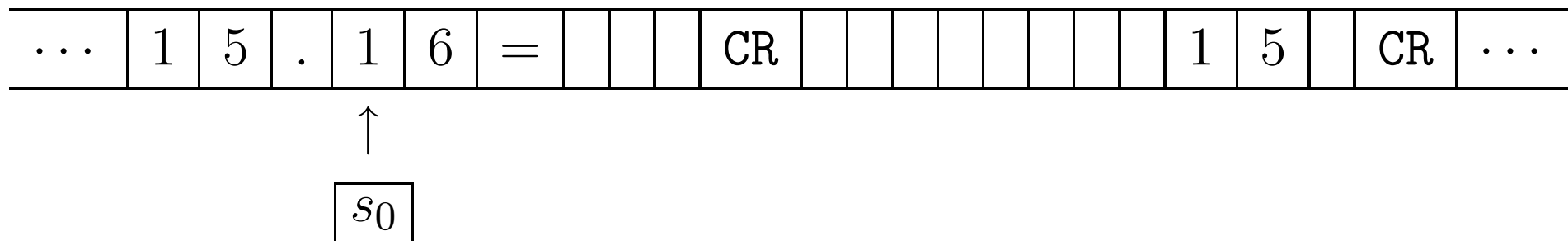


Steps in Formalising TMs

- Agent operates purely mechanistically:
Reads a symbol, and depending on it changes it and makes a movement.
Agent himself will have only finite memory.
→ There is a finite state of the agent, and, depending on the state and the symbol at the head, a next state, a new symbol, and a movement is chosen.

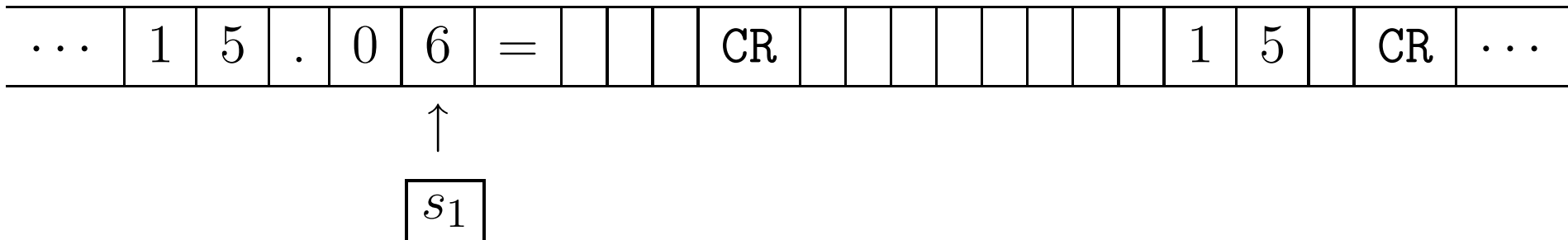
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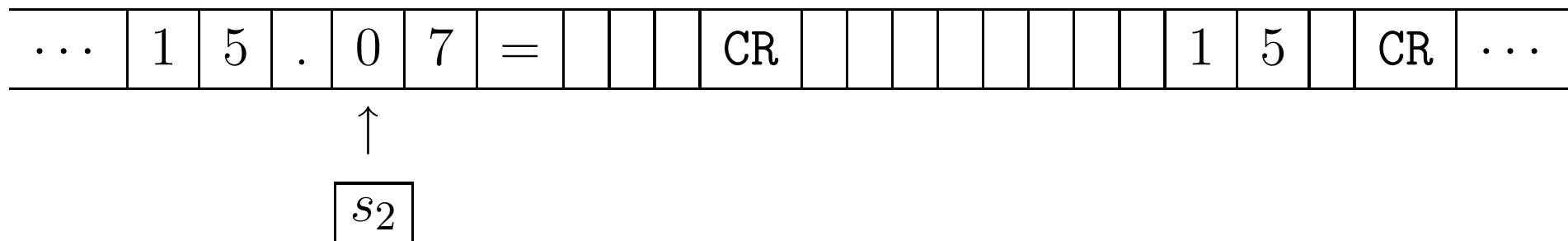
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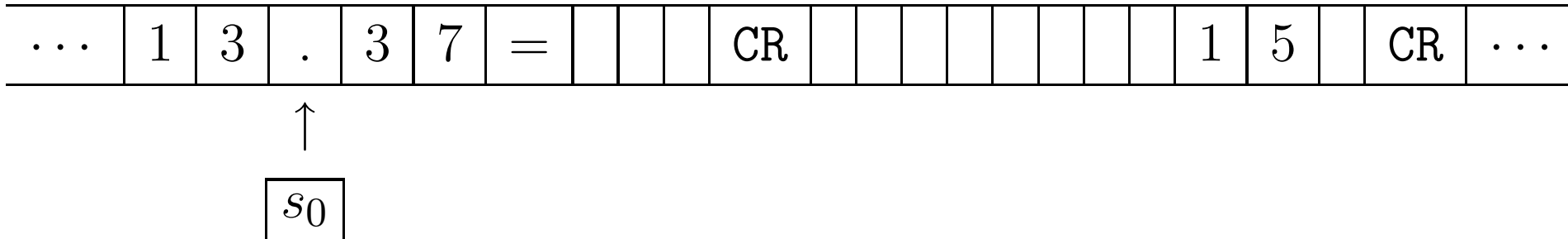
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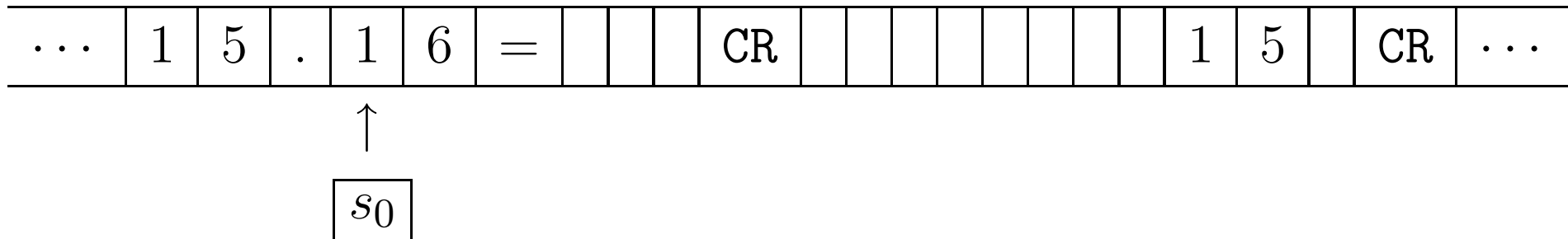


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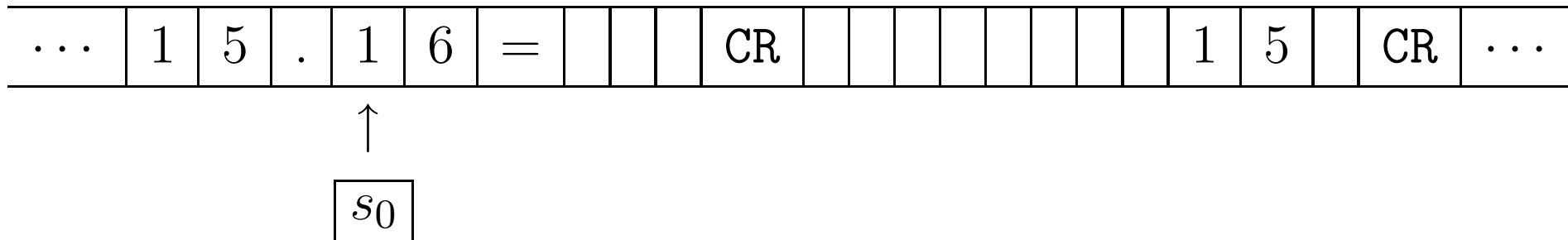


Definition of TMs



- A Turing machine is a quintuple $(\Sigma, S, l, \sqcup, s_0)$, where
 - Σ is a finite set of symbols, called the alphabet of the Turing machine. On the tape, the symbols in Σ will be written.
 - S is a finite set of states.

Definition of TMs



- I is a finite sets of quintuples (s, a, s', a', D) , where
 - $s, s' \in S$,
 - $a, a' \in \Sigma$
 - $D \in \{L, R\}$,s.t. for every $s \in S, a \in \Sigma$ there is **at most one** s', a', D s.t. $(s, a, s', a', D) \in S$.
The elements of I are called instructions.
- $\sqcup \sqcup \in \Sigma$ (a symbol for blank).
- $s_0 \in S$ (the initial state).

Meaning of Instructions

- A instruction $(s, a, s', a', D) \in I$ means the following:
 - If the Turing machine is in state s , and the symbol at position of the head is a , then
 - the state is changed to s' ,
 - the symbol at this position is changed to a' ,
 - if $D = L$, the head moves left,
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Example:

$(s_0, 1, s_1, 0, R)$

$(s_1, 6, s_2, 7, L)$

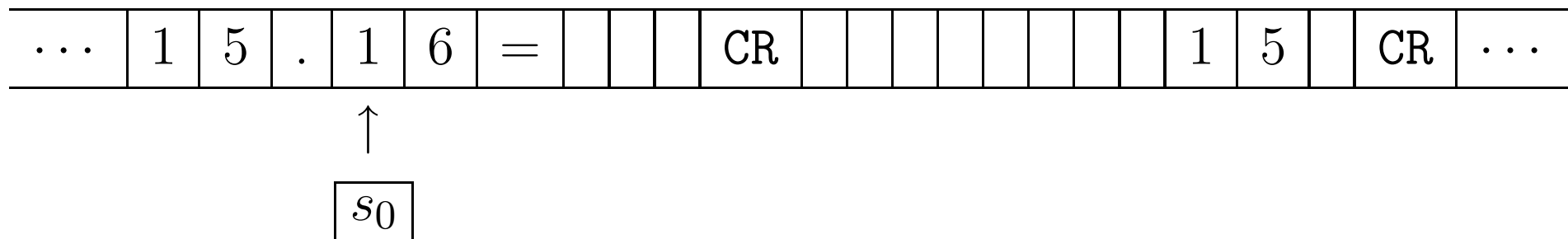
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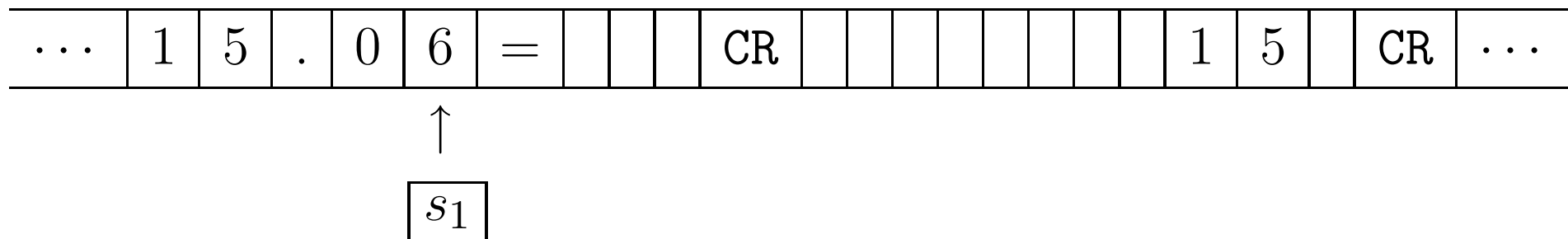
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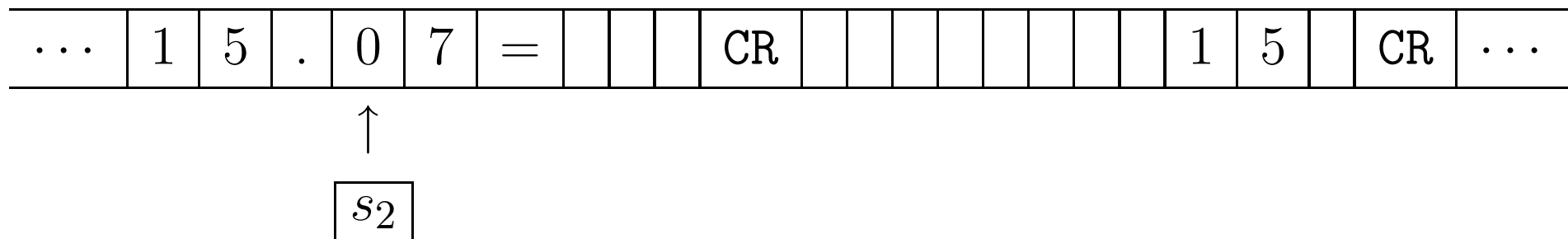
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Meaning of Instructions

- Note that for the above it is important that for every $s \in S$, $a \in \Sigma$ there is **at most one** s', a', D s.t. $(s, a, s', a', D) \in S$.
 - Without this condition, there might be more than one choice of selecting a new tape symbol, next state and direction.
 - If we omit this definition, we obtain a non-deterministic TM. In this case the machine selects in each step one of the possible choices (provided there exist one) at random.
- If the Turing machine is in a state s and reads symbol a at his head, and there are no s', a', D s.t. $(s, a, s', a', D) \in I$, then the Turing machine stops.

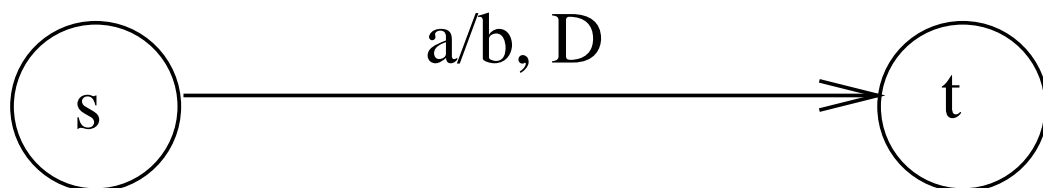
Visualisation of TMs

- A TM

$$(\Sigma, S, I, \perp, s_0)$$

can be visualised by a labelled graph as follows:

- Vertices: states (i.e. S).
- Edges: If $(s, a, t, b, D) \in I$, then there is an edge



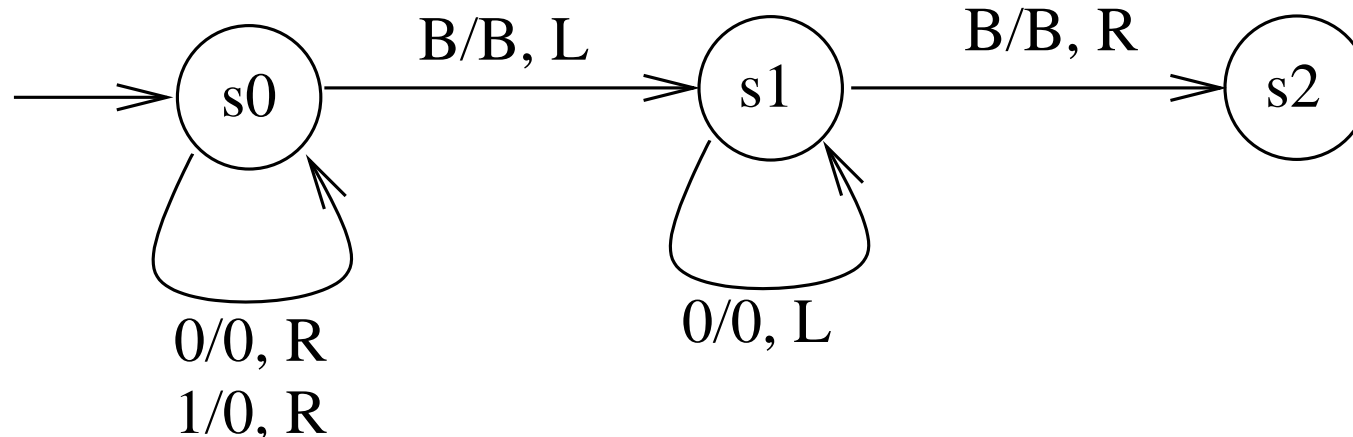
- Furthermore we write an arrow to the initial state coming from nowhere.
- If there are several vertices from s to s' , one draws only one arrow with one label for each vertex.

Example

The Turing machine with initial state s_0 and instructions

$$\begin{aligned} & \{(s_0, 0, s_0, 0, R), \\ & (s_0, 1, s_0, 0, R), \\ & (s_0, \sqcup, s_1, \sqcup, L), \\ & (s_1, 0, s_1, 0, L), \\ & (s_1, \sqcup, s_2, \sqcup, R)\} \end{aligned}$$

is visualised as follows (we write B instead of \sqcup):



Example

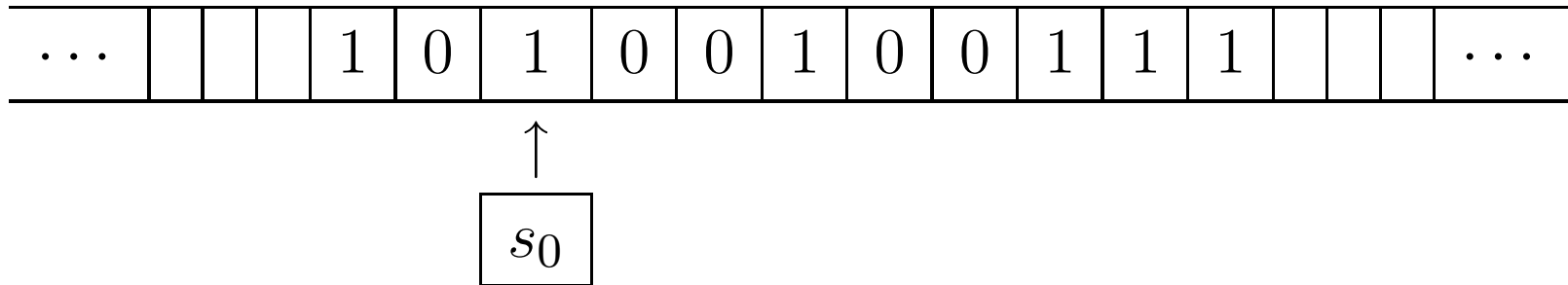
- The TM on the previous slide sets the binary number the head is pointing to to zero, provided to the left of the head there are is a blank.
- **Exercise:**
 - This example assumes that the TM points to the left most digit of a binary number.
 - Modify this TM, so that it works as well if the TM points initially to any digit of a binary number.

Example of a TM

- Development of a TM with $\Sigma = \{0, 1, \sqcup\}$,
 - where \sqcup is the symbol for the blank entry.
- Functionality of the TM:
 - Assume initially the following:
 - The tape contains binary number,
 - The rest of the tape contains \sqcup .
 - The head points to any digit of the number.
 - The TM in state s_0 .
 - Then the TM stops after finitely many steps and then
 - the tape contains the original number incremented by one,
 - the rest of tape contains \sqcup ,
 - the head points to most significant bit.

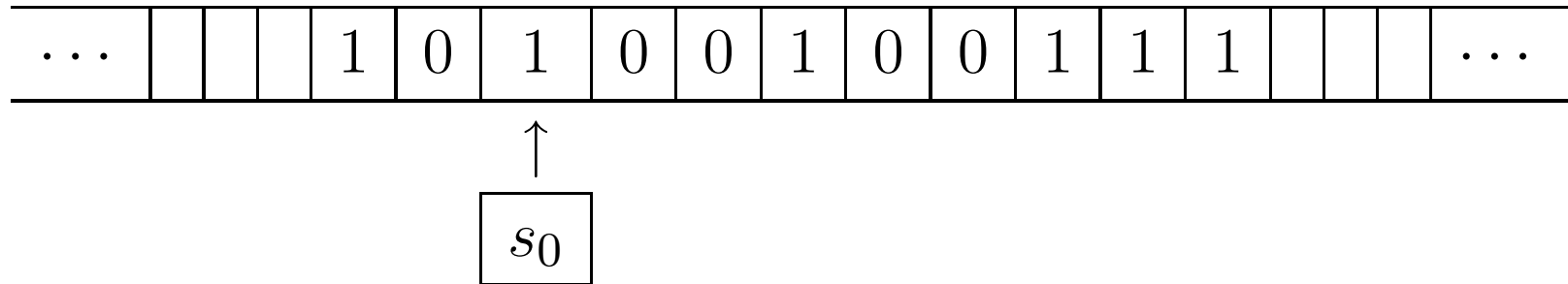
Example

Initially

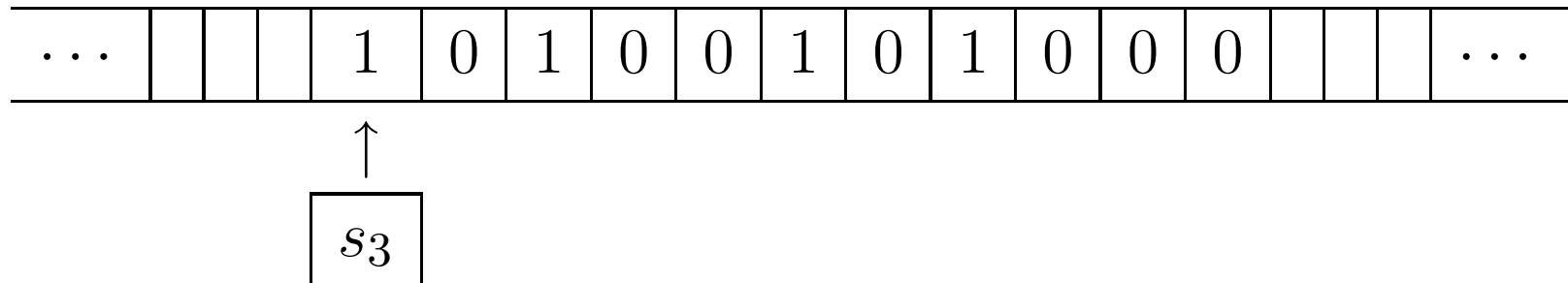


Example

Initially



Finally



Construction of the TM

- TM is $(\{0, 1, \sqcup\}, S, I, \sqcup, s_0)$.
- States S and instructions I developed in the following.

Step 1

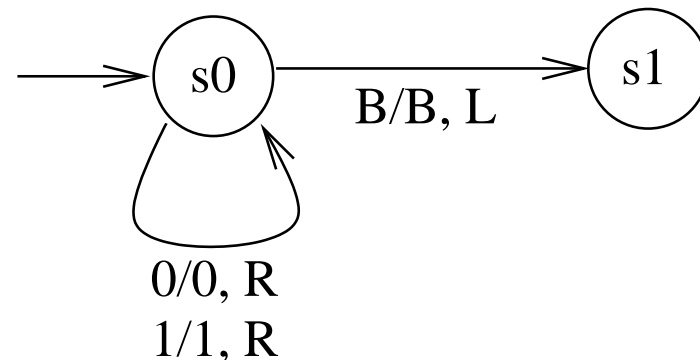
- Initially, move head to least significant bit.
 - I.e. as long as symbol at head is 0 or 1, move right, leave symbol as it is.
 - If symbol is \sqcup , move head left, leave symbol again as it is.
- Achieved by the following instructions:

$(s_0, 0, s_0, 0, R)$

$(s_0, 1, s_0, 1, R)$

$(s_0, \sqcup, s_1, \sqcup, L)$

- At the end TM is in state s_1 .



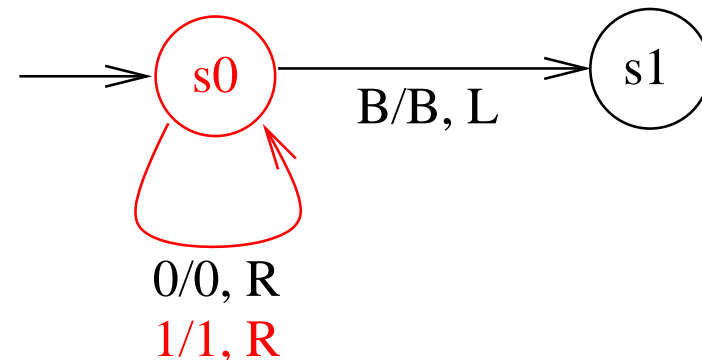
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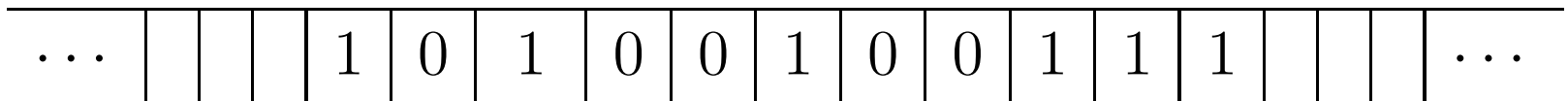
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s_0
(a)

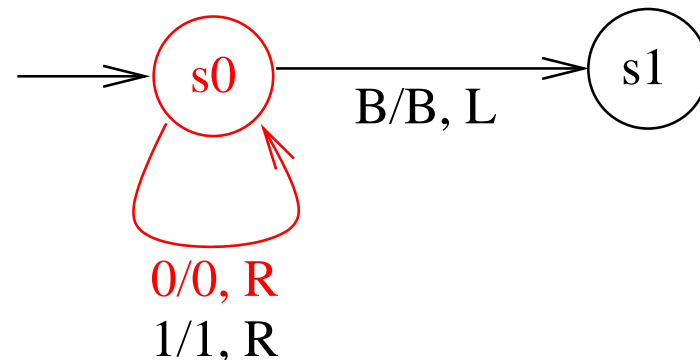
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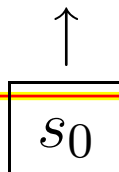
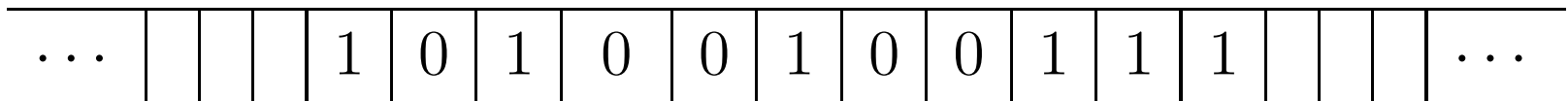
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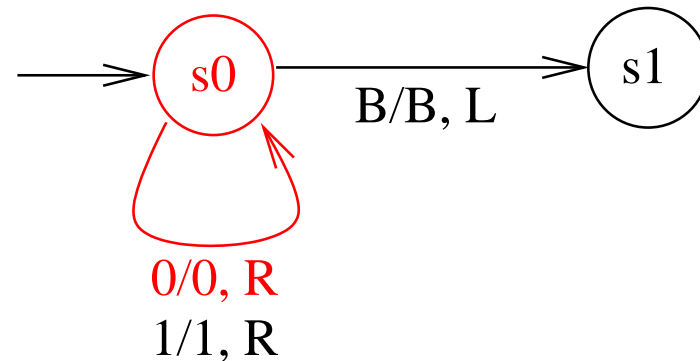
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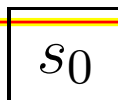
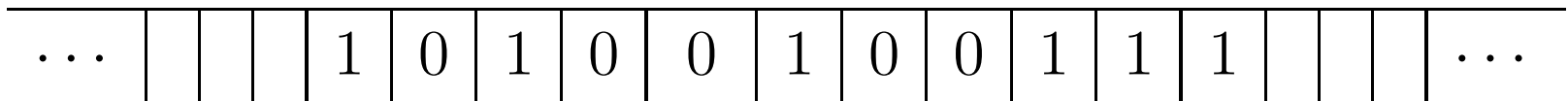
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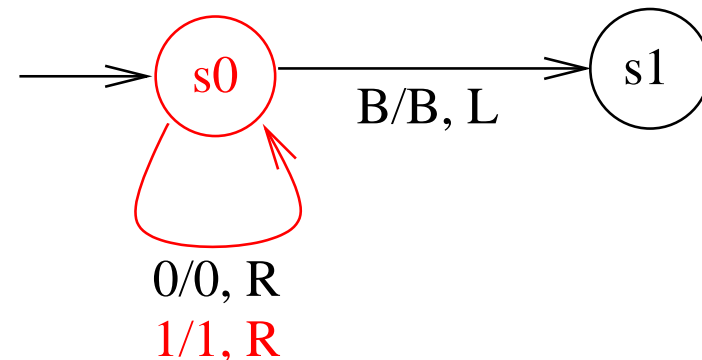
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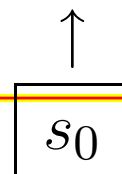
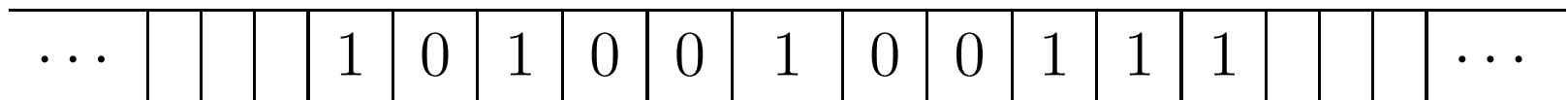
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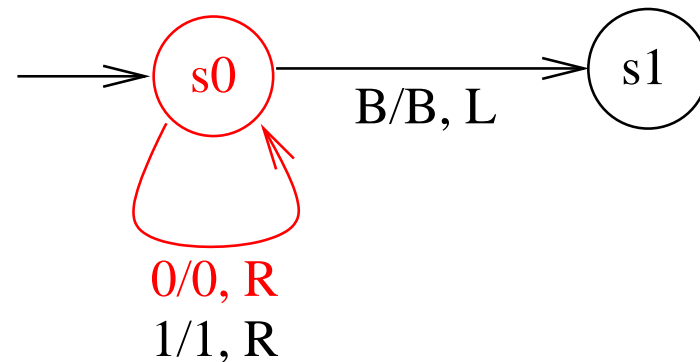
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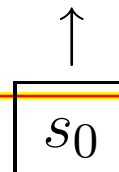
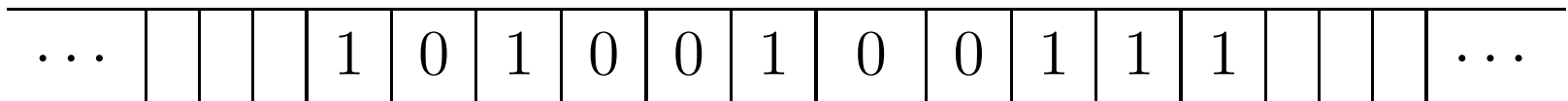
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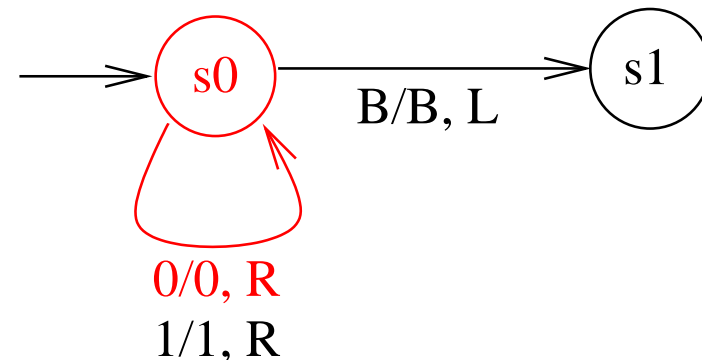
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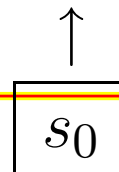
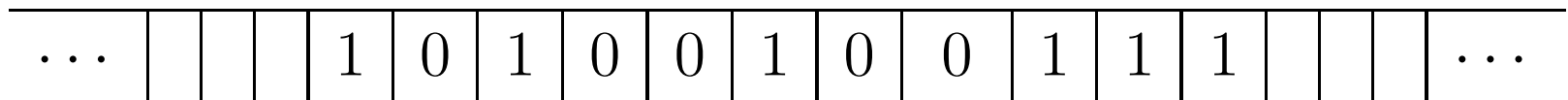
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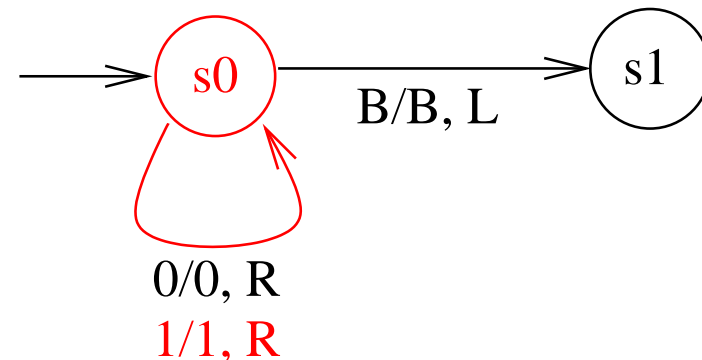
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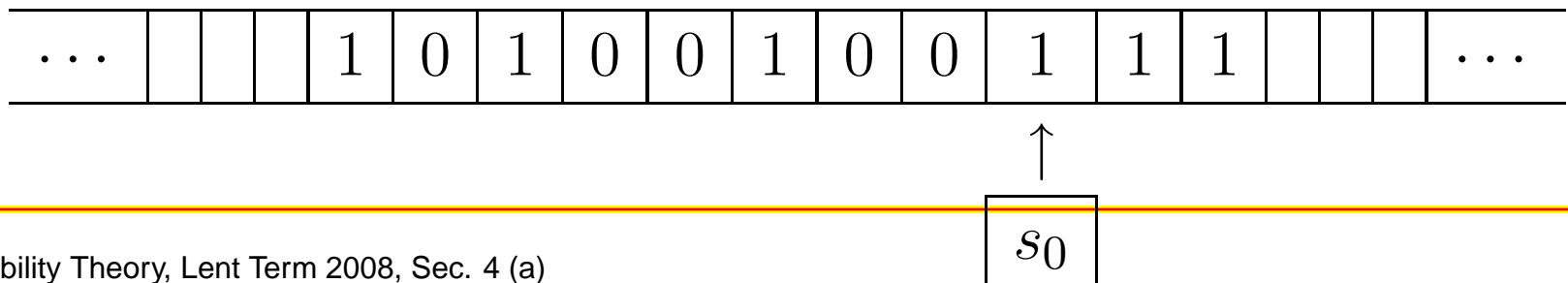
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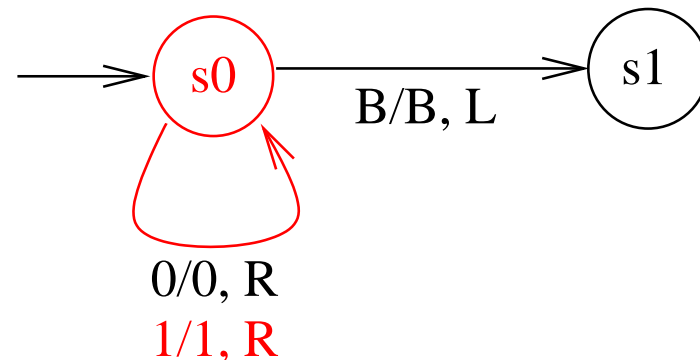
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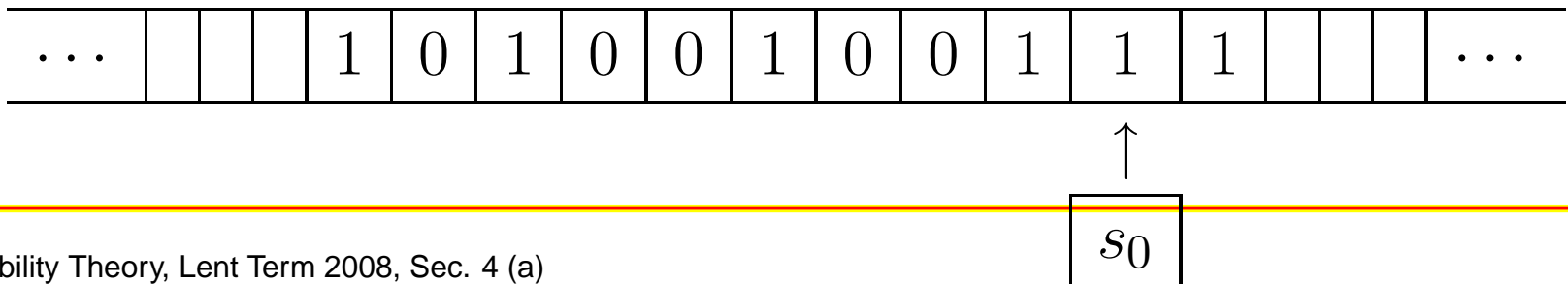
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- At the end TM is in state s_1 .



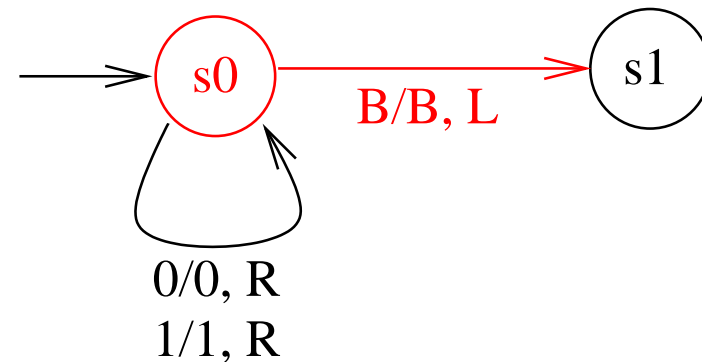
Step 1

- Initially, move head to least significant bit.
 - I.e. as long as symbol at head is 0 or 1, move right, leave symbol as it is.
 - If symbol is \sqcup , move head left, leave symbol again as it is.
- Achieved by the following instructions:

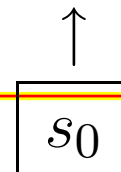
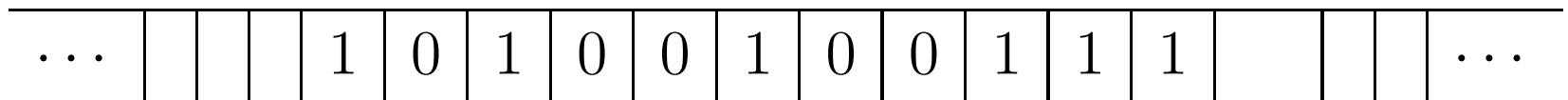
$(s_0, 0, s_0, 0, R)$

$(s_0, 1, s_0, 1, R)$

$(s_0, \sqcup, s_1, \sqcup, L)$



- At the end TM is in state s_1 .



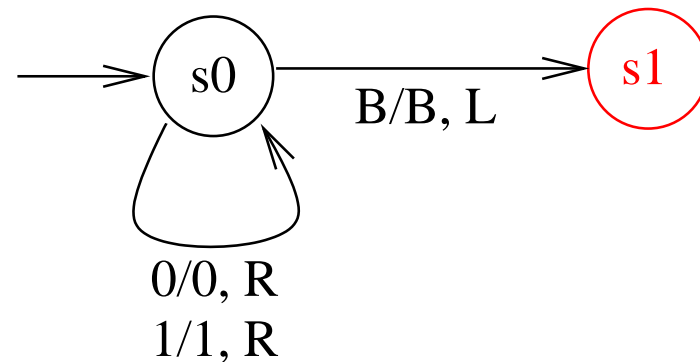
Step 1

- Initially, move head to least significant bit.
 - I.e. as long as symbol at head is 0 or 1, move right, leave symbol as it is.
 - If symbol is \perp , move head left, leave symbol again as it is.
- Achieved by the following instructions:

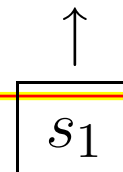
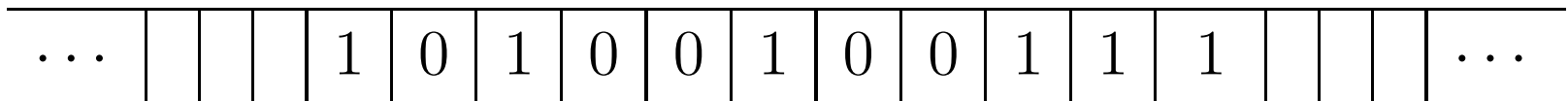
$(s_0, 0, s_0, 0, R)$

$(s_0, 1, s_0, 1, R)$

$(s_0, \perp, s_1, \perp, L)$



- At the end TM is in state s_1 .



Step 2

Increasing a binary number b done as follows:

- **Case number consists of 1 only:**

- I.e. $b = \underbrace{(111 \cdots 111)}_{k \text{ times}}_2$.

- $b + 1 = \underbrace{(1\ 000 \cdots 000)}_{k \text{ times}}_2$.

- Obtained by replacing all ones by zeros and then replacing the first blank symbol by 1.

Step 2

• Otherwise:

- Then the representation of the number contains at the end one 0 followed by ones only.
Includes case where the least significant digit is 0.
- Example 1: $b = (010001\mathbf{0111})_2$, one 0 followed by 3 ones.
- Example 2: $b = (010001001\mathbf{0})_2$, least significant digit is 0.
- Let $b = (b_0b_1 \cdots b_k \underbrace{0111 \cdots 111}_{l \text{ times}})_2$.
- $b + 1$ obtained by replacing the final block of ones by 0 and the 0 by 1:
 $b + 1 = (b_0b_1 \cdots b_k \underbrace{1000 \cdots 000}_{l \text{ times}})_2$.

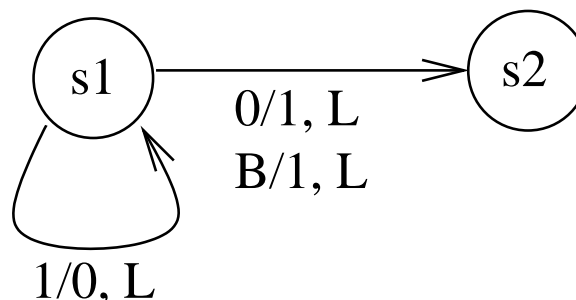
Step 2 – General Situation

- We have to replace, as long as we find ones, the ones by zeros, and move left, until we encounter a 0 or a \sqcup , which is replaced by a 1.
- So we need a new state s_2 , and the following instructions

$(s_1, 1, s_1, 0, L)$

$(s_1, 0, s_2, 1, L)$

$(s_1, \sqcup, s_2, 1, L)$



- At the end the head will be one field to the left of the 1 written, and the state will be s_2 .

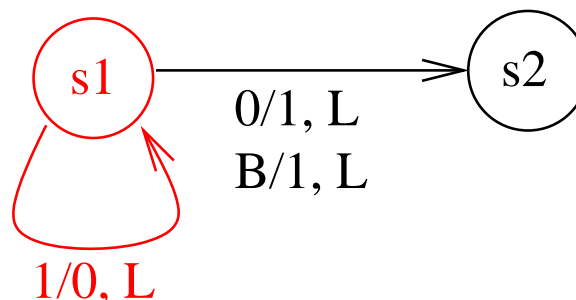
Step 2 – General Situation

- We have to replace, as long as we find ones, the ones by zeros, and move left, until we encounter a 0 or a \sqcup , which is replaced by a 1.
- So we need a new state s_2 , and the following instructions

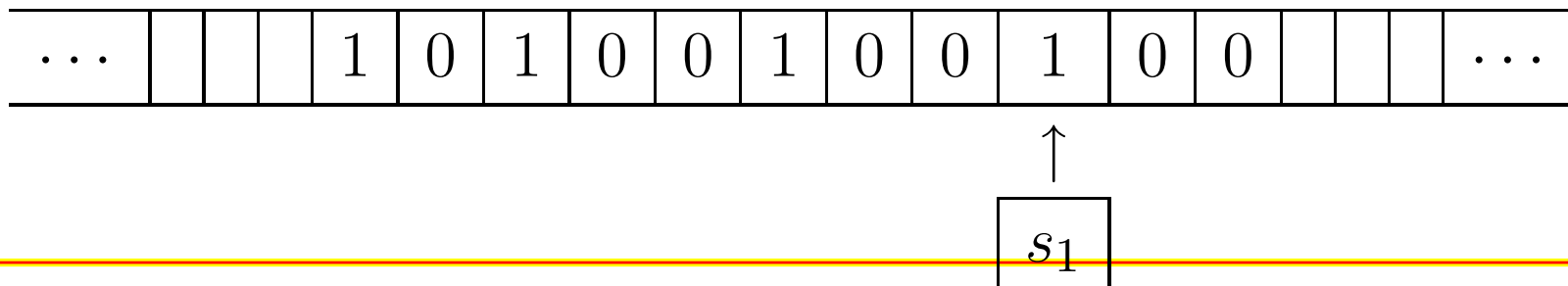
$(s_1, 1, s_1, 0, L)$

$(s_1, 0, s_2, 1, L)$

$(s_1, \sqcup, s_2, 1, L)$



- At the end the head will be one field to the left of the 1 written, and the state will be s_2 .



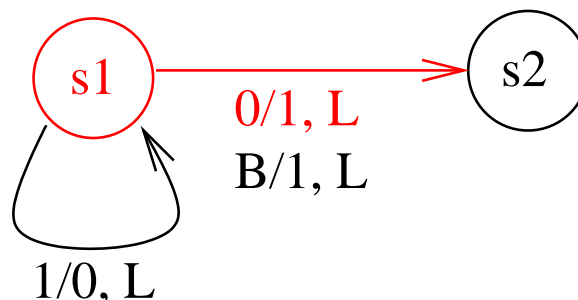
Step 2 – General Situation

- We have to replace, as long as we find ones, the ones by zeros, and move left, until we encounter a 0 or a \sqcup , which is replaced by a 1.
- So we need a new state s_2 , and the following instructions

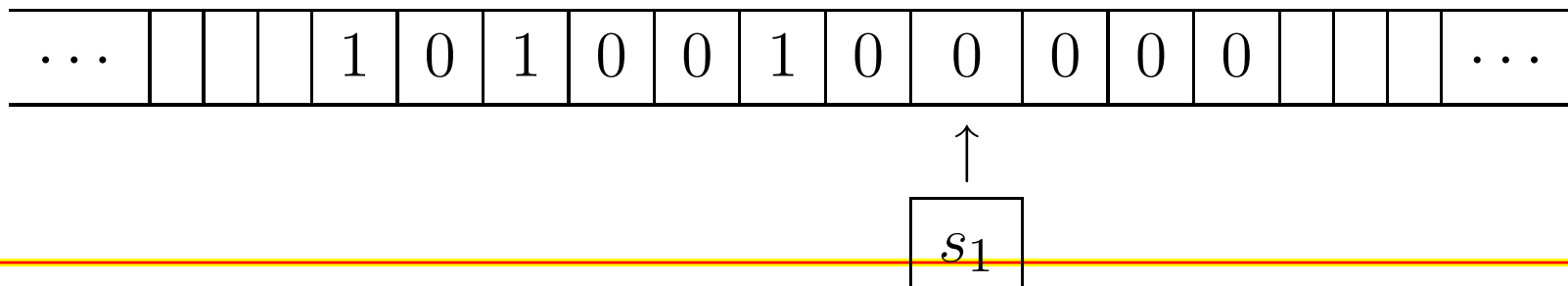
$(s_1, 1, s_1, 0, L)$

$(s_1, 0, s_2, 1, L)$

$(s_1, \sqcup, s_2, 1, L)$



- At the end the head will be one field to the left of the 1 written, and the state will be s_2 .



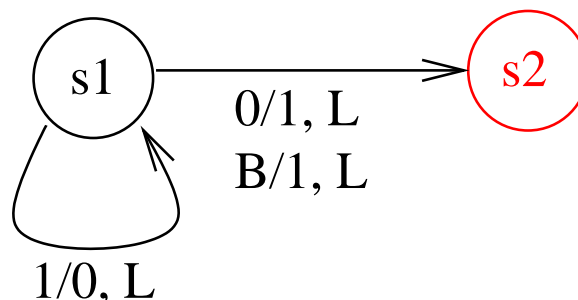
Step 2 – General Situation

- We have to replace, as long as we find ones, the ones by zeros, and move left, until we encounter a 0 or a \sqcup , which is replaced by a 1.
- So we need a new state s_2 , and the following instructions

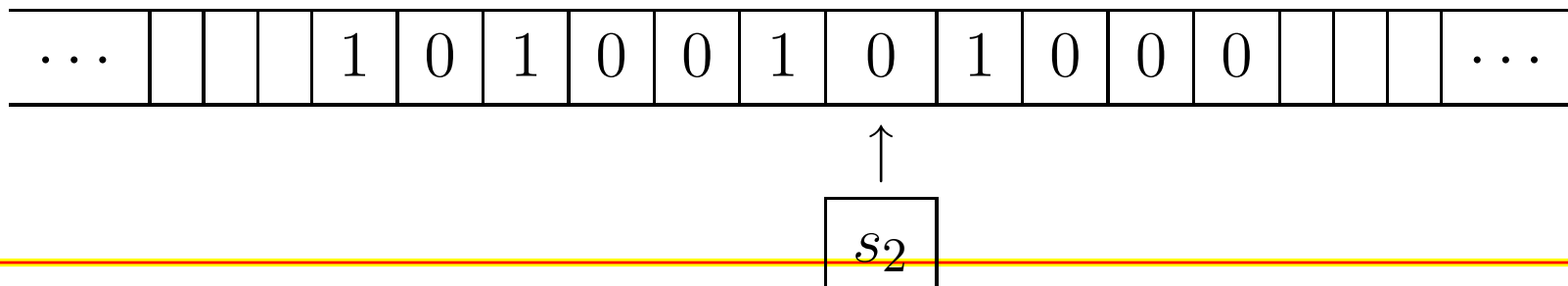
$(s_1, 1, s_1, 0, L)$

$(s_1, 0, s_2, 1, L)$

$(s_1, \sqcup, s_2, 1, L)$



- At the end the head will be one field to the left of the 1 written, and the state will be s_2 .



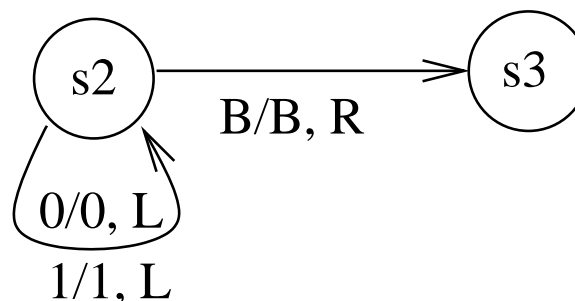
Step 3

Finally, we have to move the most significant bit, which is done as follows

$(s_2, 0, s_2, 0, L)$

$(s_2, 1, s_2, 1, L)$

$(s_2, \sqcup, s_3, \sqcup, R)$



The program terminates in state s_3 .

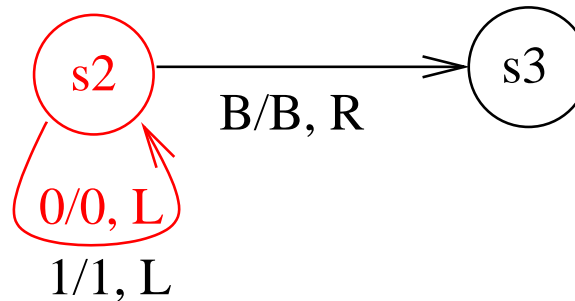
Step 3

Finally, we have to move the most significant bit, which is done as follows

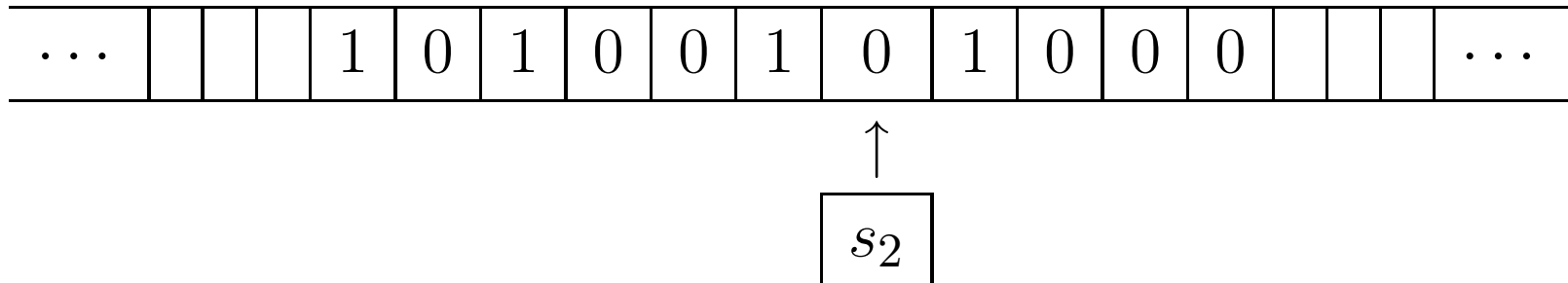
$(s_2, 0, s_2, 0, L)$

$(s_2, 1, s_2, 1, L)$

$(s_2, \perp, s_3, \perp, R)$



The program terminates in state s_3 .



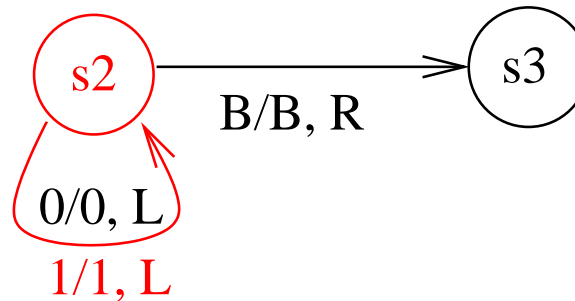
Step 3

Finally, we have to move the most significant bit, which is done as follows

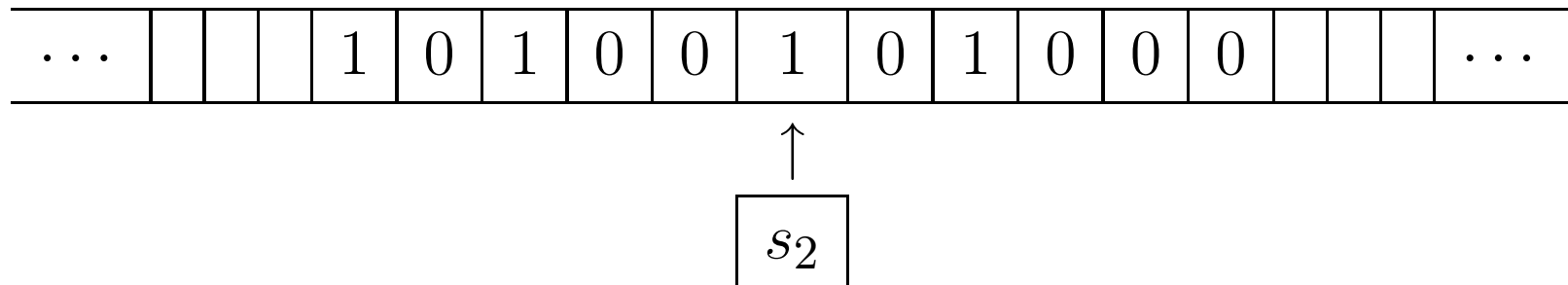
$(s_2, 0, s_2, 0, L)$

$(s_2, 1, s_2, 1, L)$

$(s_2, \perp, s_3, \perp, R)$



The program terminates in state s_3 .



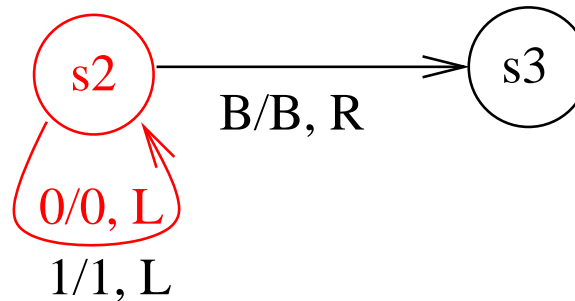
Step 3

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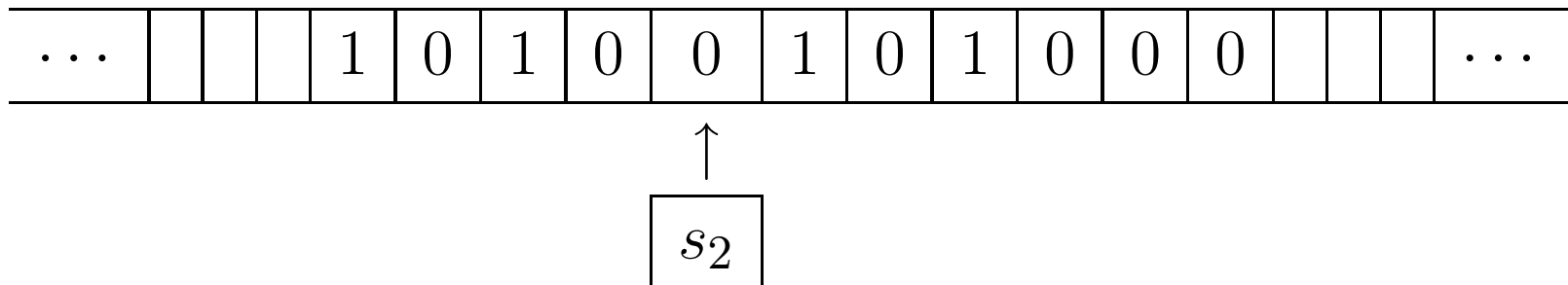
$(s_2, 0, s_2, 0, L)$

$(s_2, 1, s_2, 1, L)$

$(s_2, \perp, s_3, \perp, R)$



The program terminates in state s_3 .



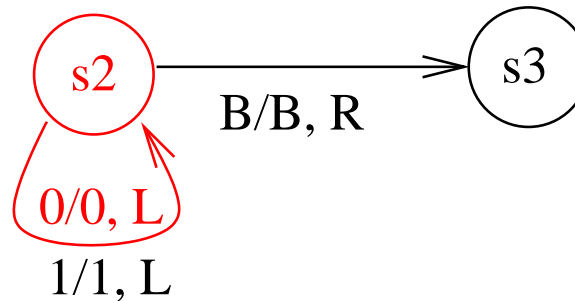
Step 3

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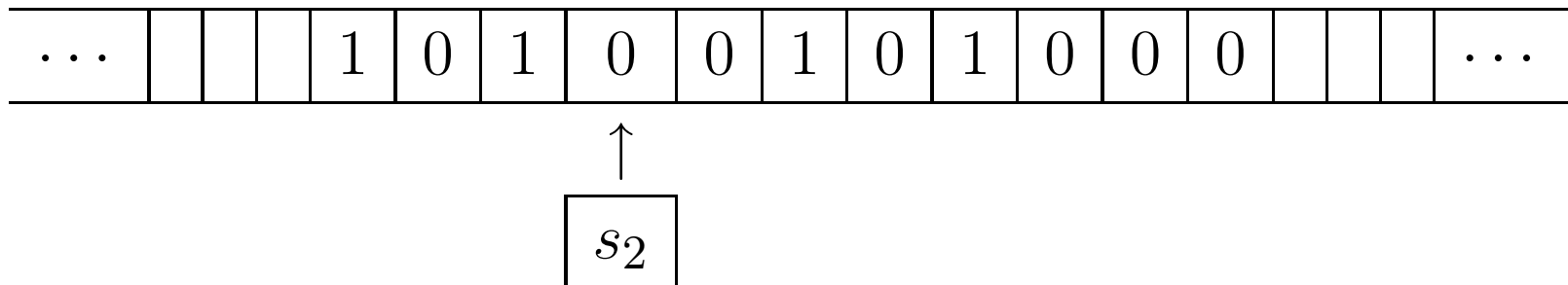
$(s_2, 0, s_2, 0, L)$

$(s_2, 1, s_2, 1, L)$

$(s_2, \sqcup, s_3, \sqcup, R)$



The program terminates in state s_3 .



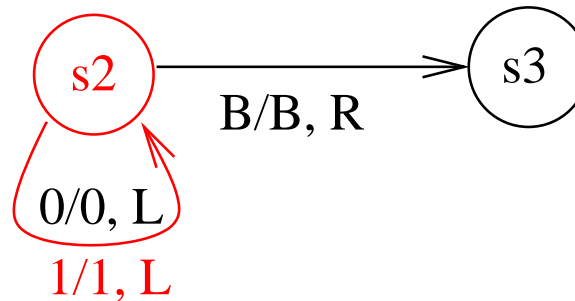
Step 3

Finally, we have to move the most significant bit, which is done as follows

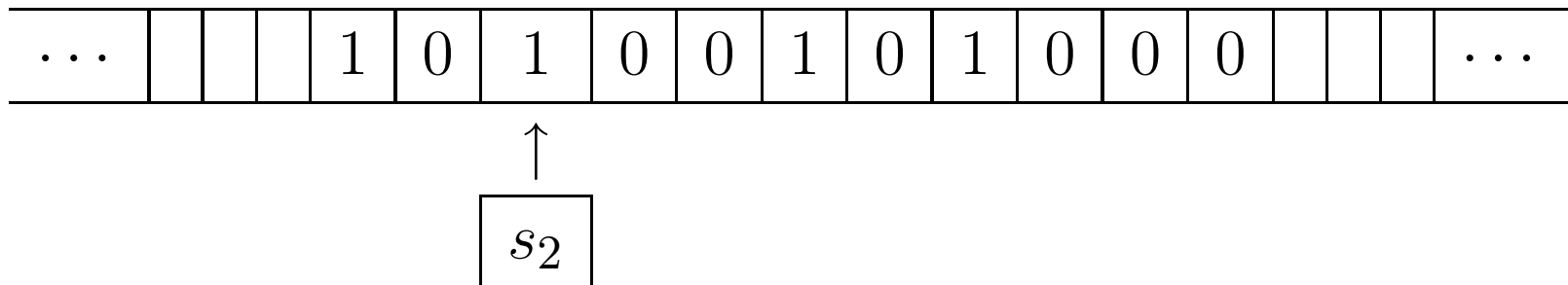
$(s_2, 0, s_2, 0, L)$

$(s_2, 1, s_2, 1, L)$

$(s_2, \sqcup, s_3, \sqcup, R)$



The program terminates in state s_3 .



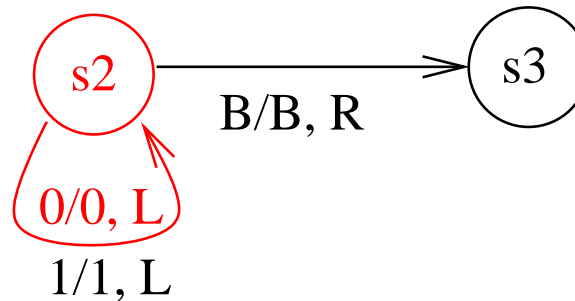
Step 3

Finally, we have to move the most significant bit, which is done as follows

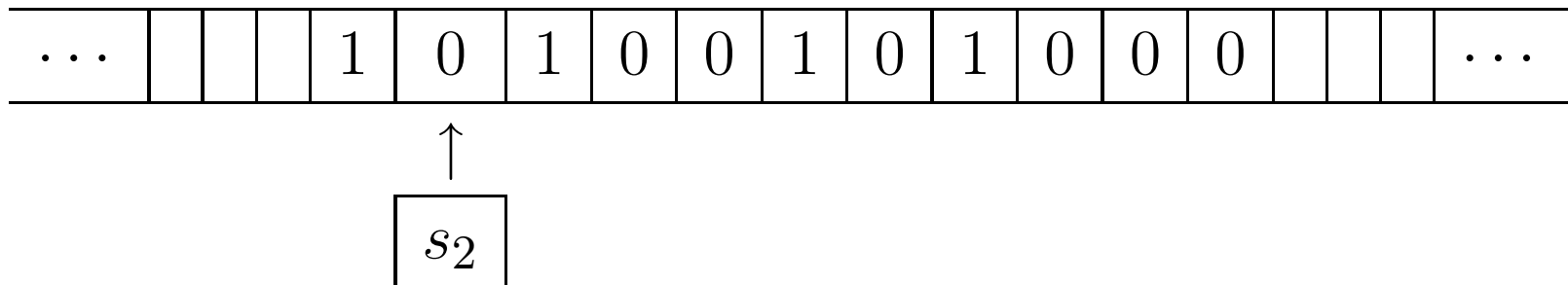
$(s_2, 0, s_2, 0, L)$

$(s_2, 1, s_2, 1, L)$

$(s_2, \perp, s_3, \perp, R)$



The program terminates in state s_3 .



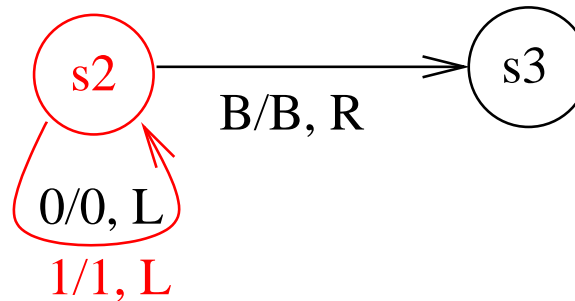
Step 3

Finally, we have to move the most significant bit, which is done as follows

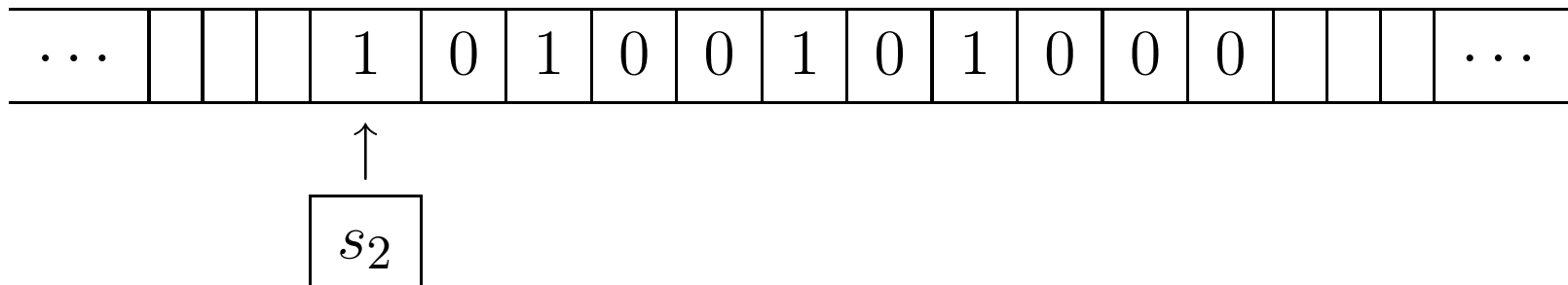
$(s_2, 0, s_2, 0, L)$

$(s_2, 1, s_2, 1, L)$

$(s_2, \perp, s_3, \perp, R)$



The program terminates in state s_3 .



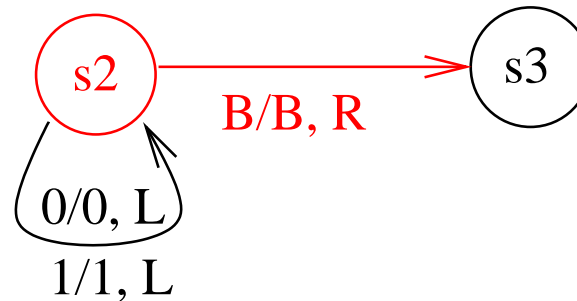
Step 3

Finally, we have to move the most significant bit, which is done as follows

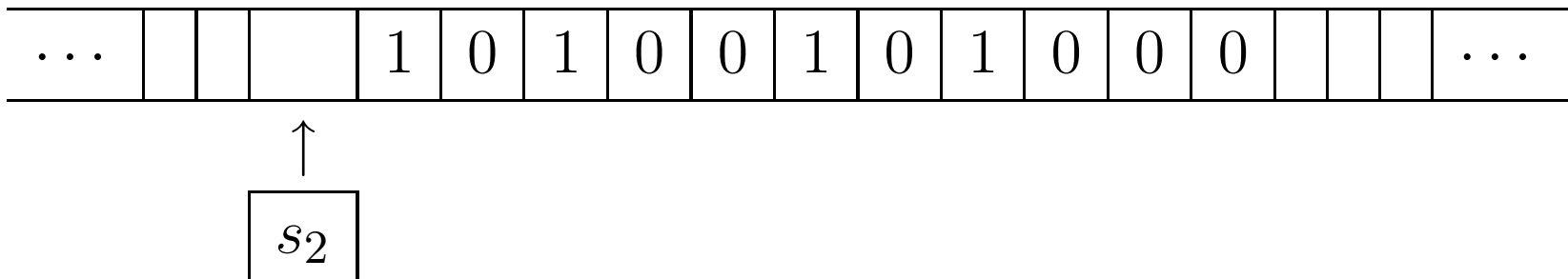
$(s_2, 0, s_2, 0, L)$

$(s_2, 1, s_2, 1, L)$

$(s_2, \perp, s_3, \perp, R)$



The program terminates in state s_3 .



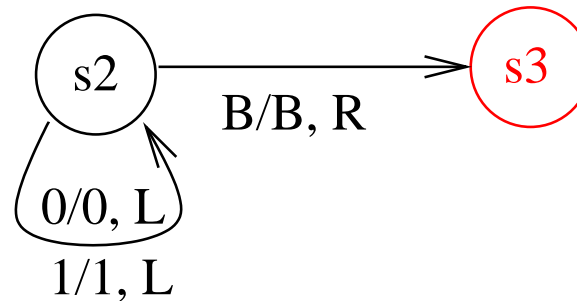
Step 3

Finally, we have to move the most significant bit, which is done as follows

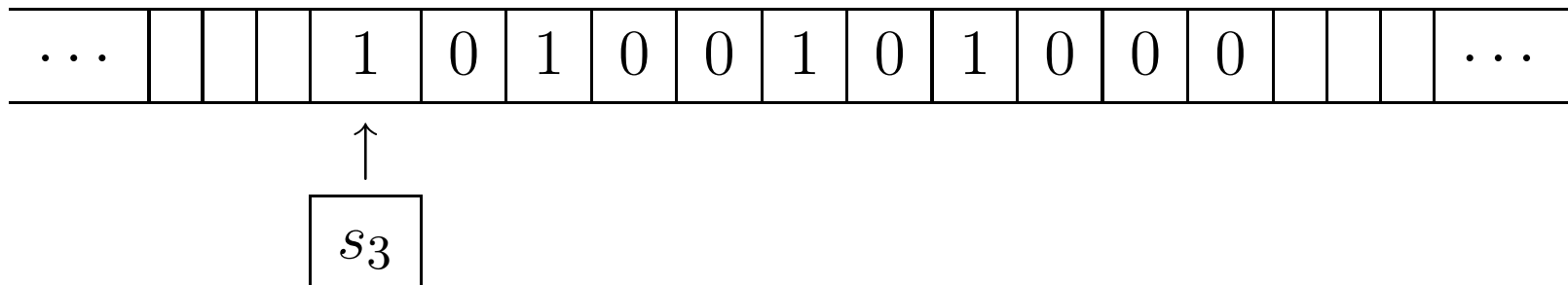
$(s_2, 0, s_2, 0, L)$

$(s_2, 1, s_2, 1, L)$

$(s_2, \sqcup, s_3, \sqcup, R)$



The program terminates in state s_3 .

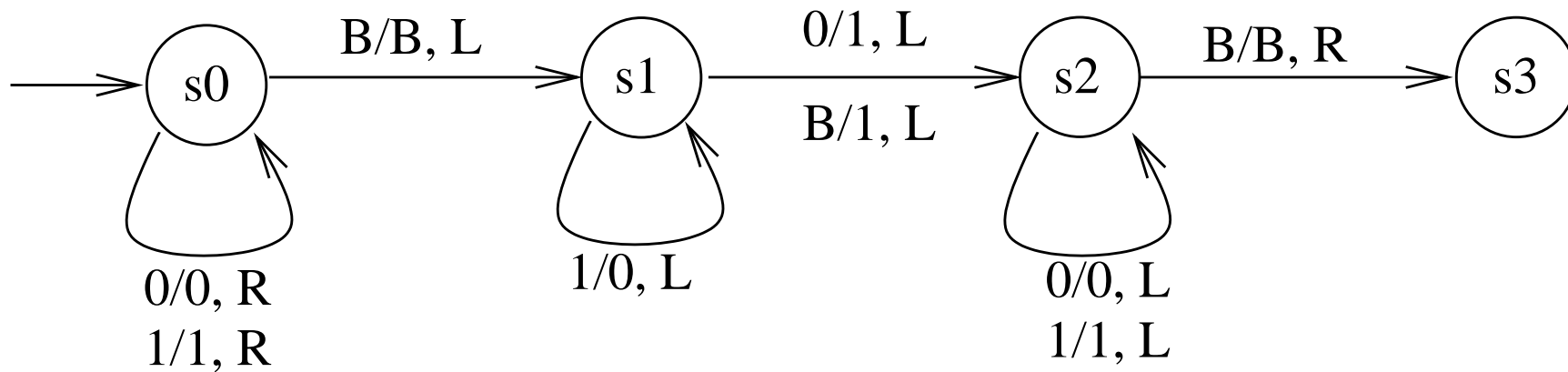


Complete TM

The complete TM is as follows:

$$\begin{aligned} &(\{0, 1, \perp\}, \\ &\{s_0, s_1, s_2, s_3\}, \\ &\{(s_0, 0, s_0, 0, R), \\ &\quad (s_0, 1, s_0, 1, R), \\ &\quad (s_0, \perp, s_1, \perp, L), \\ &\quad (s_1, 1, s_1, 0, L), \\ &\quad (s_1, 0, s_2, 1, L), \\ &\quad (s_1, \perp, s_2, 1, L), \\ &\quad (s_2, 0, s_2, 0, L), \\ &\quad (s_2, 1, s_2, 1, L), \\ &\quad (s_2, \perp, s_3, \perp, R)\}, s_0, \perp) \end{aligned}$$

Complete TM



Notation: bin

- TMs usually operate on binary numbers.
- Therefore we define for a natural number $\text{bin}(n)$ as the sequence of digits representing the unique standard binary representation of n .
 - So $\text{bin}(n)$ has no leading zeros, except for $\text{bin}(0) := "0"$.
- Examples:
 - $\text{bin}(0) = "0"$,
 - $\text{bin}(1) = "1"$,
 - $\text{bin}(2) = "10"$,
 - $\text{bin}(3) = "11"$,
 - $\text{bin}(4) = "100"$, etc.

Notation: $\widetilde{\text{bin}}$

- In order to read off the final result, we need to interpret an arbitrary finite sequence of 0, 1 as a binary number, even if it has leading zeros.
- We define $\widetilde{\text{bin}}(n)$ as one of the possible binary representations of n , allowing leading 0.
 - So $\widetilde{\text{bin}}(1)$ can be "1", "01", "001", etc.
 - In the special case 0 we treat the empty string as one of the possible representations, so $\widetilde{\text{bin}}(0)$ can be "", "0", "00", "000", etc.

Notation: $\widetilde{\text{bin}}$

- When carrying out intermediate calculations, it is easier to refer to $\widetilde{\text{bin}}(n)$ rather than $\text{bin}(n)$
 - E.g. we can set a number on the tape easily to an element of $\widetilde{\text{bin}}(0)$ by overwriting it with 0s.
 - In order to set it to $\text{bin}(0)$ one would need to make sure that exactly one 0 remains. Then one usually has to shift left the content of the tape to the right of the original number.

Definition 4.1

Let $T = (\Sigma, S, l, \sqcup, s_0)$ be a Turing machine with $\{0, 1\} \subseteq \Sigma$.

Define for every $k \in \mathbb{N}$ $\mathcal{T}^{(k)} : \mathbb{N}^k \xrightarrow{\sim} \mathbb{N}$, where

$\mathcal{T}^{(k)}(a_0, \dots, a_{k-1})$ is computed as follows:

- **Initialisation:**

- We write on the tape

$\text{bin}(a_0)\sqcup\text{bin}(a_1)\sqcup\cdots\sqcup\text{bin}(a_{k-1})$.

- E.g. if $k = 3$, $a_0 = 0$, $a_1 = 3$, $a_2 = 2$ then we write

$0\sqcup 11\sqcup 10$.

- All other cells contain \sqcup .
- The head is at the left most bit of the arguments written on the tape.
- The state is set to s_0 .

- **Iteration:** Run the TM, until it stops.

Definition 4.1

● Output:

- **Case 1:** The TM stops.

Only finitely many cells are non-blank.

Let tape, starting from the head-position, contain $b_0b_1 \cdots b_{k-1}c$ where $b_i \in \{0, 1\}$ and $c \notin \{0, 1\}$.

(k might be 0).

Let

$$a = (b_0, \dots, b_{k-1})_2 .$$

(in case $k = 0$, $a = 0$).

This means " $b_0 \cdots b_{k-1}$ " is one of the choices for $\widetilde{\text{bin}}(a)$.

Then

$$T^{(k)}(a_0, \dots, a_{k-1}) \simeq a .$$

Definition 4.1

Example: Let $\Sigma = \{0, 1, a, b, \sqcup\}$ where $0, 1, a, b, \sqcup$ are different.

- If the tape starting with the head is as follows:

- $01010\sqcup 0101\sqcup$

- or $01010a\sqcup$,

output is $(01010)_2 = (10)_{10}$.

- If tape starting with the head is as follows:

- $ab\sqcup$

- or a ,

- or \sqcup ,

the output is 0.

- **Case 2:** Otherwise.

Then $T^{(k)}(a_0, \dots, a_{k-1}) \uparrow$.

Definition 4.2

$f : \mathbb{N}^k \xrightarrow{\sim} \mathbb{N}$ is Turing-computable, in short TM-computable, if $f = T^{(k)}$ for some TM T , the alphabet of which contains $\{0, 1\}$.

Example: That $\text{succ} : \mathbb{N} \xrightarrow{\sim} \mathbb{N}$ and $\text{zero} : \mathbb{N} \xrightarrow{\sim} \mathbb{N}$ are Turing-computable was shown above.

Remark

- If the tape of the Turing machine initially contains only finitely many cells which are not blank, then at any step during the execution of the TM only finitely many cells are non blank.
 - Follows since in each step at most one cell can be modified to become non-blank.
 - So in finitely many steps only finitely many cells can be converted from blank to non-blank.

(b) URM Comput. \Rightarrow TM Comput.

Theorem 4.3 If $f : \mathbb{N}^n \xrightarrow{\sim} \mathbb{N}$ is URM-computable, then it is as well Turing-computable by a TM with alphabet $\{0, 1, \sqcup\}$.

Proof of Theorem 4.3

Notation

The tape of a TM contains a_0, \dots, a_l means:

- Starting from the head position, the cells of the tape contain a_0, \dots, a_l .
- All other cells contain \sqcup .

Proof of Theorem 4.3

Assume

- $f = U^{(n)}$,
- U refers only to R_0, \dots, R_{l-1} and $l > n$,

We define a TM T , which simulates U . Done as follows:

- That the registers R_0, \dots, R_{l-1} contain a_0, \dots, a_{l-1} is simulated by the tape containing $\widetilde{\text{bin}}(a_0) \sqcup \dots \sqcup \widetilde{\text{bin}}(a_{l-1})$.
- An instruction I_j will be simulated by states $s_{j,0}, \dots, s_{j,i}$ with instructions for those states.

Conditions on the Simulation

- Assume the URM U is in a state s.t.
 - R_0, \dots, R_{l-1} contain a_0, \dots, a_{l-1} ,
 - the URM is about to execute I_j .
- Assume after executing I_j , the URM is in a state where
 - R_0, \dots, R_{l-1} contain b_0, \dots, b_{l-1} ,
 - the PC contains k .
- Then we want that, if configuration of the TM T is, s.t.
 - the tape contains $\widetilde{\text{bin}}(a_0) \sqcup \widetilde{\text{bin}}(a_1) \sqcup \dots \sqcup \widetilde{\text{bin}}(a_{l-1})$,
 - and the TM is in state $s_{j,0}$,
- then the TM reaches a configuration s.t.
 - the tape contains $\widetilde{\text{bin}}(b_0) \sqcup \widetilde{\text{bin}}(b_1) \sqcup \dots \sqcup \widetilde{\text{bin}}(b_{l-1})$,
 - the TM is in state $s_{k,0}$.

Example

- Assume the URM is about to execute instruction
 - $I_4 = \text{pred}(2)$ (i.e. $PC = 4$),
 - with register contents

R_0	R_1	R_2
2	1	3

- Then the URM will end with
 - $PC = 5$
 - and register contents

R_0	R_1	R_2
2	1	2

Example

- Then we want that, if the simulating TM is
 - in state $s_{4,0}$,
 - with tape content $\widetilde{\text{bin}}(2)_L_\sqcup\widetilde{\text{bin}}(1)_L_\sqcup\widetilde{\text{bin}}(3)$
- it should reach
 - state $s_{5,0}$
 - with tape content $\widetilde{\text{bin}}(2)_L_\sqcup\widetilde{\text{bin}}(1)_L_\sqcup\widetilde{\text{bin}}(2)$

Proof of Theorem 4.3

- Furthermore, we need initial states $s_{\text{init},0}, \dots, s_{\text{init},j}$ and corresponding instructions, s.t.
 - if the TM initially contains

$$\widetilde{\text{bin}}(b_0)_ \sqcup \widetilde{\text{bin}}(b_1)_ \sqcup \dots \sqcup \widetilde{\text{bin}}(b_{n-1})_$$

- it will reach state $s_{0,0}$ with the tape containing

$$\widetilde{\text{bin}}(b_0)_ \sqcup \widetilde{\text{bin}}(b_1)_ \sqcup \dots \sqcup \widetilde{\text{bin}}(b_{n-1})_ \sqcup \underbrace{0_ \sqcup 0_ \sqcup \dots \sqcup 0_}_{l-n \text{ times}}$$

Proof of Theorem 4.3

Assume the run of the URM, starting with R_i containing $a_{0,i} = a_i$ $i = 0, \dots, n - 1$, and $a_{0,i} = 0$ for $i = n, \dots, l - 1$ is as follows:

Instruction	R_0	R_1	\dots	R_{n-1}	R_n	\dots	R_{l-1}
I_0	a_0	a_1	\dots	a_{n-1}	0	\dots	0
=	=	=		=	=		=
I_{k_0}	$a_{0,0}$	$a_{0,1}$	\dots	$a_{0,n-1}$	$a_{0,n}$	\dots	$a_{0,l-1}$
I_{k_1}	$a_{1,0}$	$a_{1,1}$	\dots	$a_{1,n-1}$	$a_{1,n}$	\dots	$a_{1,l-1}$
I_{k_2}	$a_{2,0}$	$a_{2,1}$	\dots	$a_{2,n-1}$	$a_{2,n}$	\dots	$a_{2,l-1}$
	\dots						

Proof of Theorem 4.3

Then the corresponding TM will successively reach the following configurations:

State Tape contains

$s_{\text{init},0}$ $\widetilde{\text{bin}}(a_0)\sqcup\widetilde{\text{bin}}(a_1)\sqcup\cdots\sqcup\widetilde{\text{bin}}(a_{n-1})\sqcup$

$s_{0,0}$ $\widetilde{\text{bin}}(a_0)\sqcup\widetilde{\text{bin}}(a_1)\sqcup\cdots\sqcup\widetilde{\text{bin}}(a_{n-1})\sqcup\widetilde{\text{bin}}(0)\sqcup\cdots\sqcup\widetilde{\text{bin}}(0)\sqcup$

=

$s_{k_0,0}$ $\widetilde{\text{bin}}(a_{0,0})\sqcup\widetilde{\text{bin}}(a_{0,1})\sqcup\cdots\sqcup\widetilde{\text{bin}}(a_{0,l-1})\sqcup$

$s_{k_1,0}$ $\widetilde{\text{bin}}(a_{1,0})\sqcup\widetilde{\text{bin}}(a_{1,1})\sqcup\cdots\sqcup\widetilde{\text{bin}}(a_{1,l-1})\sqcup$

$s_{k_2,0}$ $\widetilde{\text{bin}}(a_{2,0})\sqcup\widetilde{\text{bin}}(a_{2,1})\sqcup\cdots\sqcup\widetilde{\text{bin}}(a_{2,l-1})\sqcup$

...

Example

Consider the URM program U (which was discussed already in the section on URMs):

$$I_0 = \text{ifzero}(0, 3)$$

$$I_1 = \text{pred}(0)$$

$$I_2 = \text{ifzero}(1, 0)$$

$$U^{(1)}(a) \simeq 0.$$

Example

$I_0 = \text{ifzero}(0, 3)$

$I_1 = \text{pred}(0)$

$I_2 = \text{ifzero}(1, 0)$

Example

$I_0 = \text{ifzero}(0, 3)$

$I_1 = \text{pred}(0)$

$I_2 = \text{ifzero}(1, 0)$

A run of $U^{(1)}(2)$ is as follows:

Example

$I_0 = \text{ifzero}(0, 3)$

$I_1 = \text{pred}(0)$

$I_2 = \text{ifzero}(1, 0)$

A run of $U^{(1)}(2)$ is as follows:

Instruction	R_0	R_1
-------------	-------	-------

Example

$I_0 = \text{ifzero}(0, 3)$

$I_1 = \text{pred}(0)$

$I_2 = \text{ifzero}(1, 0)$

A run of $U^{(1)}(2)$ is as follows:

Instruction	R_0	R_1
I_0	2	0

Example

$I_0 = \text{ifzero}(0, 3)$

$I_1 = \text{pred}(0)$

$I_2 = \text{ifzero}(1, 0)$

A run of $U^{(1)}(2)$ is as follows:

Instruction	R_0	R_1
I_0	2	0
I_1	2	0

Example

$I_0 = \text{ifzero}(0, 3)$

$I_1 = \text{pred}(0)$

$I_2 = \text{ifzero}(1, 0)$

A run of $U^{(1)}(2)$ is as follows:

Instruction	R_0	R_1
I_0	2	0
I_1	2	0
I_2	1	0

Example

$I_0 = \text{ifzero}(0, 3)$

$I_1 = \text{pred}(0)$

$I_2 = \text{ifzero}(1, 0)$

A run of $U^{(1)}(2)$ is as follows:

Instruction	R_0	R_1
I_0	2	0
I_1	2	0
I_2	1	0
I_0	1	0

Example

$I_0 = \text{ifzero}(0, 3)$

$I_1 = \text{pred}(0)$

$I_2 = \text{ifzero}(1, 0)$

A run of $U^{(1)}(2)$ is as follows:

Instruction	R_0	R_1
I_0	2	0
I_1	2	0
I_2	1	0
I_0	1	0
I_1	1	0

Example

$I_0 = \text{ifzero}(0, 3)$

$I_1 = \text{pred}(0)$

$I_2 = \text{ifzero}(1, 0)$

A run of $U^{(1)}(2)$ is as follows:

Instruction	R_0	R_1
I_0	2	0
I_1	2	0
I_2	1	0
I_0	1	0
I_1	1	0
I_2	0	0

Example

$I_0 = \text{ifzero}(0, 3)$

$I_1 = \text{pred}(0)$

$I_2 = \text{ifzero}(1, 0)$

A run of $U^{(1)}(2)$ is as follows:

Instruction	R_0	R_1
I_0	2	0
I_1	2	0
I_2	1	0
I_0	1	0
I_1	1	0
I_2	0	0
I_0	0	0

Example

$I_0 = \text{ifzero}(0, 3)$

$I_1 = \text{pred}(0)$

$I_2 = \text{ifzero}(1, 0)$

A run of $U^{(1)}(2)$ is as follows:

Instruction	R_0	R_1
I_0	2	0
I_1	2	0
I_2	1	0
I_0	1	0
I_1	1	0
I_2	0	0
I_0	0	0
I_3	0	0

Example

$I_0 = \text{ifzero}(0, 3)$

$I_1 = \text{pred}(0)$

$I_2 = \text{ifzero}(1, 0)$

A run of $U^{(1)}(2)$ is as follows:

Instruction	R_0	R_1
I_0	2	0
I_1	2	0
I_2	1	0
I_0	1	0
I_1	1	0
I_2	0	0
I_0	0	0
I_3	0	0

URM Stops

Corresponding TM Simulation

$I_0 = \text{ifzero}(0, 3)$

$I_1 = \text{pred}(0)$

$I_2 = \text{ifzero}(1, 0)$

Instruction	R_0	R_1	State of TM	Content of Tape
			$s_{\text{init},0}$	$\widetilde{\text{bin}}(2) \sqcup \sqcup$
I_0	2	0	$s_{0,0}$	$\widetilde{\text{bin}}(2) \sqcup \sqcup \widetilde{\text{bin}}(0) \sqcup \sqcup$
I_1	2	0	$s_{1,0}$	$\widetilde{\text{bin}}(2) \sqcup \sqcup \widetilde{\text{bin}}(0) \sqcup \sqcup$
I_2	1	0	$s_{2,0}$	$\widetilde{\text{bin}}(1) \sqcup \sqcup \widetilde{\text{bin}}(0) \sqcup \sqcup$
I_0	1	0	$s_{0,0}$	$\widetilde{\text{bin}}(1) \sqcup \sqcup \widetilde{\text{bin}}(0) \sqcup \sqcup$
I_1	1	0	$s_{1,0}$	$\widetilde{\text{bin}}(1) \sqcup \sqcup \widetilde{\text{bin}}(0) \sqcup \sqcup$
I_2	0	0	$s_{2,0}$	$\widetilde{\text{bin}}(0) \sqcup \sqcup \widetilde{\text{bin}}(0) \sqcup \sqcup$
I_0	0	0	$s_{0,0}$	$\widetilde{\text{bin}}(0) \sqcup \sqcup \widetilde{\text{bin}}(0) \sqcup \sqcup$
I_3	0	0	$s_{3,0}$	$\widetilde{\text{bin}}(0) \sqcup \sqcup \widetilde{\text{bin}}(0) \sqcup \sqcup$

URM Stops

TM Stops

Proof of Theorem 4.3

If we have defined this we have

• If

$$U^n(a_0, \dots, a_{n-1}) \downarrow ,$$
$$U^n(a_0, \dots, a_{n-1}) \simeq c ,$$

then U eventually stops with R_i containing some values b_i , where $b_0 = c$.

Then, the TM T starting with

$$\text{bin}(a_0) \sqcup \dots \sqcup \text{bin}(a_{n-1})$$

will eventually terminate in a configuration

$$\widetilde{\text{bin}}(b_0) \sqcup \dots \sqcup \widetilde{\text{bin}}(b_{k-1}) .$$

for some $k \geq n$.

Therefore $T^n(a_0, \dots, a_{n-1}) \simeq b_0 = c$.

Proof of Theorem 4.3

• If

$$U^n(a_0, \dots, a_{n-1}) \uparrow ,$$

the URM U will loop and the TM T will carry out the same steps as the URM and loop as well.

Therefore

$$T^n(a_0, \dots, a_{n-1}) \uparrow ,$$

again

$$U^n(a_0, \dots, a_{n-1}) \simeq T^n(a_0, \dots, a_{n-1}) .$$

Proof of Theorem 4.3

- It follows

$$U^{(n)} = T^{(n)} ,$$

and the proof is complete, if the simulation has been introduced.

- The following slides contain a detailed proof, which will not be presented in the lecture this year.
[Jump over remaining proof.](#)

Proof of Theorem 4.3

Informal description of the simulation of URM instructions.

- **Initialisation.**

Initially, the tape contains $\text{bin}(a_0) \sqcup \cdots \sqcup \text{bin}(a_{n-1})$.

We need to obtain configuration:

$$\widetilde{\text{bin}}(a_0) \sqcup \cdots \sqcup \widetilde{\text{bin}}(a_{n-1}) \sqcup \underbrace{\widetilde{\text{bin}}(0) \sqcup \cdots \sqcup \widetilde{\text{bin}}(0)}_{l - n \text{ times}}.$$

Achieved by

- moving head to the end of the initial configuration
- inserting, starting from the next blank, $l - n$ -times $0 \sqcup$,
- then moving back to the beginning.

Proof of Theorem 4.3

- **Simulation of URM instructions.**

- **Simulation of instruction $I_k = \text{succ}(j)$.**

Need to increase $(j + 1)$ st binary number by 1 Initial configuration:

$\widetilde{\text{bin}}(c_0) \sqcup \widetilde{\text{bin}}(c_1) \sqcup \cdots \sqcup \widetilde{\text{bin}}(c_j) \sqcup \cdots \sqcup \widetilde{\text{bin}}(c_l) \sqcup$
 \uparrow
 $s_{k,0}$

- First move to the $(j + 1)$ st blank to the right. Then we are at the end of the $(j + 1)$ st binary number.

$\widetilde{\text{bin}}(c_0) \sqcup \widetilde{\text{bin}}(c_1) \sqcup \cdots \sqcup \widetilde{\text{bin}}(c_j) \sqcup \cdots \sqcup \widetilde{\text{bin}}(c_l) \sqcup$
 \uparrow

Proof of Theorem 4.3

$\widetilde{\text{bin}}(c_0) \sqcup \widetilde{\text{bin}}(c_1) \sqcup \cdots \sqcup \widetilde{\text{bin}}(c_j) \sqcup \cdots \sqcup \widetilde{\text{bin}}(c_l) \sqcup \cdots$

\uparrow

- Now perform the operation for increasing by 1 as above.

At the end we obtain:

$\widetilde{\text{bin}}(c_0) \sqcup \widetilde{\text{bin}}(c_1) \sqcup \cdots \sqcup \widetilde{\text{bin}}(c_j + 1) \sqcup \cdots \sqcup \widetilde{\text{bin}}(c_l) \sqcup \cdots$

\uparrow

- It might be that we needed to write over the separating blank a 1, in which case we have:

$\widetilde{\text{bin}}(c_0) \sqcup \widetilde{\text{bin}}(c_1) \sqcup \cdots \widetilde{\text{bin}}(c_{j-1}) \widetilde{\text{bin}}(c_j + 1) \sqcup \cdots \sqcup \widetilde{\text{bin}}(c_l) \sqcup \cdots$

\uparrow

Proof of Theorem 4.3

- In the latter case, shift all symbols to the left once left, in order to obtain a separating \sqcup between the l th and $l - 1$ st entry.

We obtain

$$\widetilde{\text{bin}}(c_0) \sqcup \widetilde{\text{bin}}(c_1) \sqcup \cdots \widetilde{\text{bin}}(c_{j-1}) \sqcup \widetilde{\text{bin}}(c_j + 1) \sqcup \cdots$$

↑

- Otherwise, move the head to the left, until we reach the $(j + 1)$ st blank to the left, and then move it once to the right.

We obtain

$$\widetilde{\text{bin}}(c_0) \sqcup \widetilde{\text{bin}}(c_1) \sqcup \cdots \sqcup \widetilde{\text{bin}}(c_j + 1) \sqcup \cdots \sqcup \widetilde{\text{bin}}(c_j)$$

↑

Proof of Theorem 4.3

- **Simulation of instruction** $I_k = \text{pred}(j)$.
 - Assume the configuration at the beginning is :

$$\widetilde{\text{bin}}(c_0) \sqcup \widetilde{\text{bin}}(c_1) \sqcup \cdots \sqcup \widetilde{\text{bin}}(c_j) \sqcup \cdots \sqcup \widetilde{\text{bin}}(c_l)$$

↑

We want to achieve

$$\widetilde{\text{bin}}(c_0) \sqcup \widetilde{\text{bin}}(c_1) \sqcup \cdots \sqcup \widetilde{\text{bin}}(c_j - 1) \sqcup \cdots \sqcup \widetilde{\text{bin}}(c_l)$$

↑

Done as follows:

Proof of Theorem 4.3

Initially: $\widetilde{\text{bin}}(c_0) \sqcup \cdots \sqcup \widetilde{\text{bin}}(c_j) \sqcup \cdots \sqcup \widetilde{\text{bin}}(c_l)$

↑

Finally: $\widetilde{\text{bin}}(c_0) \sqcup \cdots \sqcup \widetilde{\text{bin}}(c_j \div 1) \sqcup \cdots \sqcup \widetilde{\text{bin}}(c_l)$

↑

- Move to end of the $(j + 1)$ st number.
- Check, if the number consists only of zeros or not.
 - If it consists only of zeros, $\text{pred}(j)$ doesn't change anything.
 - Otherwise, number is of the form $b_0 \cdots b_k 1 \underbrace{00 \cdots 0}_{l' \text{ times}}$.

Replace it by $b_0 \cdots b_k 0 \underbrace{11 \cdots 1}_{l' \text{ times}}$.

Done as for succ.

Proof of Theorem 4.3

Initially: $\widetilde{\text{bin}}(c_0) \sqcup \dots \sqcup \widetilde{\text{bin}}(c_j) \sqcup \dots \sqcup \widetilde{\text{bin}}(c_l)$

↑

Finally: $\widetilde{\text{bin}}(c_0) \sqcup \dots \sqcup \widetilde{\text{bin}}(c_j \div 1) \sqcup \dots \sqcup \widetilde{\text{bin}}(c_l)$

↑

We have achieved

$\widetilde{\text{bin}}(c_0) \sqcup \widetilde{\text{bin}}(c_1) \sqcup \dots \sqcup \widetilde{\text{bin}}(c_j \div 1) \sqcup \dots \sqcup \widetilde{\text{bin}}(c_l)$

↑

Move back to the beginning:

$\widetilde{\text{bin}}(c_0) \sqcup \widetilde{\text{bin}}(c_1) \sqcup \dots \sqcup \widetilde{\text{bin}}(c_j \div 1) \sqcup \dots \sqcup \widetilde{\text{bin}}(c_l)$

↑

Proof of Theorem 4.3

- **Simulation of instruction** $I_k = \text{ifzero}(j, k')$.
 - Move to $j + 1$ st binary number on the tape.
 - Check whether it contains only zeros.
 - If yes, switch to state $s_{k',0}$.
 - Otherwise switch to state $s_{k+1,0}$.

This completes the simulation of the URM U .

Remark

- We will later show that all TM-computable functions are URM-computable.
 - This will be done by showing that
 - all TM-computable functions are partial recursive,
 - all partial recursive functions are URM-computable.
 - This will be easier than showing directly that TM-computable functions are URM-computable.
- Therefore the set of TM-computable functions and the set of URM-computable functions coincide.

Extension to Arbitrary Alphabets

- Let A be a finite alphabet s.t. $\sqcup \notin A$, and $B := A^*$.
- To a Turing machine $T = (\Sigma, S, l, \sqcup, s_0)$ with $A \subseteq \Sigma$ corresponds a partial function $T^{(A,n)} : B^n \rightsquigarrow B$, where $T^{(A,n)}(a_0, \dots, a_{n-1})$ is computed as follows:
 - Initially write $a_0 \sqcup \dots \sqcup a_{n-1}$ on the tape, otherwise \sqcup .
Start in state s_0 on the left most position of a_0 .
 - Iterate TM as before.
 - In case of termination, the output of the function is $c_0 \dots c_{l-1}$, if the tape contains, starting with the head position $c_0 \dots c_{l-1} d$ with $c_i \in A$, $d \notin A$.
 - Otherwise, the function value is undefined.

Extension to Arbitrary Alphabets

- Notion is modulo encoding of A^* into \mathbb{N} equivalent to the notion of Turing-computability on \mathbb{N} .
- However, when considering complexity bounds, this notation might be more appropriate.
 - Avoids encoding/decoding into \mathbb{N} .

Turing-Computable Predicates

- A predicate A is Turing-decidable, iff χ_A is Turing-computable.
- Instead of simulating χ_A
 - means to write the output of χ_A (a binary number 0 or 1) on the tape

it is more convenient, to take TM with two additional special states s_{true} and s_{false} corresponding to truth and falsity of the predicate.

Turing-Computable Predicates

- Then a predicate is Turing decidable, if, when we write initially the inputs as before on the tape and start executing the TM,
 - it always terminates in s_{true} or s_{false} ,
 - and it terminates in s_{true} , iff the predicate holds for the inputs,
 - and in s_{false} , otherwise.
- The latter notion is equivalent to the first notion.
- Usually the latter one is taken as basis for complexity considerations.

(c) Undecid. of the Halting Problem

- Undecidability of the Halting Problem first proved 1936 by Alan Turing.
- In this Section, we will identify computable with Turing-computable.
 - This will later be justified by the Church-Turing thesis.

History of Computability Theory



Alan Mathison Turing (1912 – 1954)

Introduced the Turing machine.
Proved the undecidability
of the Turing-Halting problem.

Definition 4.4

- (a) A problem is an n -ary predicate $M(\vec{x})$ of natural numbers, i.e. a property of n -tuples of natural numbers.
- (b) A problem (or predicate) M is (Turing-)decidable, if **the characteristic function χ_M of M** is (Turing-)computable.

Characteristic function

- Reminder:

$$\chi_M(\vec{x}) := \begin{cases} 1 & \text{if } M(\vec{x}) \text{ holds,} \\ 0 & \text{otherwise} \end{cases}$$

- If we treat true as 1 and false as 0, then the characteristic function is nothing but the Boolean valued function which decides whether $M(\vec{x})$ holds or not:

$$\chi_M(\vec{x}) = \begin{cases} \text{true} & \text{if } M(\vec{x}) \text{ holds,} \\ \text{false} & \text{otherwise} \end{cases}$$

Example of Decidable Problems

- The binary predicate

$$\text{Multiple}(x, y) :\Leftrightarrow x \text{ is a multiple of } y$$

is a predicate and therefore a problem.

- $\chi_{\text{Multiple}}(x, y)$ decides, whether $\text{Multiple}(x, y)$ holds (then it returns 1 for yes), or not:

$$\chi_{\text{Multiple}}(x, y) = \begin{cases} 1 & \text{if } x \text{ is a multiple of } y, \\ 0 & \text{if } x \text{ is not a multiple of } y. \end{cases}$$

- χ_{Multiple} is intuitively computable, therefore Multiple is decidable.

Encoding of Turing Machines

- A Turing Machine is a quintuple $(\Sigma, S, l, \perp, s_0)$.
- We can assume that \perp , each symbol of the alphabet, and each state can be represented by a string of letters and numbers.
- Then this quintuple can be written as a string of ASCII-symbols.
- \Rightarrow Turing machines can be represented as elements of A^* , where $A =$ set of ASCII-symbols.
- \Rightarrow Turing machines can be encoded as natural numbers.
 - Of course more efficient encoding exist.

Encoding of Turing Machines

- Let for a Turing machine T , $\text{encode}(T) \in \mathbb{N}$ be its code.
- It is intuitively decidable, whether a string of ASCII symbols is a Turing machine.
 - One can show that this can be decided by a Turing machine.
- \Rightarrow It is intuitively decidable, whether $n = \text{encode}(T)$ for a Turing machine T .

$\{e\}^k(n)$

- Assume $e \in \mathbb{N}$. We define a partial function $\{e\}^k : \mathbb{N}^k \rightrightarrows \mathbb{N}$, by

$$\{e\}^k(\vec{x}) \simeq \begin{cases} m & \text{if } e = \text{encode}(T) \text{ for some Turing machine } T \\ & \text{and } T^{(k)}(\vec{x}) \simeq m, \\ \perp & \text{otherwise.} \end{cases}$$

- So if $e = \text{encode}(T)$, $\{e\}^k = T^{(k)}$.
 - Roughly speaking, $\{e\}^k$ is the function computed by the e th Turing machine.
 - So for every computable (more precisely Turing-computable) function $f : \mathbb{N}^k \rightrightarrows \mathbb{N}$ there exists an e s.t. $f = \{e\}^k$.

$\{e\}^k$

- The notation $\{e\}^k$ is due to Stephen Kleene.
- $\{\}$ are called **Kleene-Brackets**.
- We write $\{e\}$ for $\{e\}^1$.

Stephen Cole Kleene



Stephen Cole Kleene (1909 – 1994)

Probably the most influential computability theorist up to now. Introduced the partial recursive functions.

The Halting Problem

- **Definition 4.5**

The Halting Problem is the following binary predicate:

$$\text{Halt}(e, n) :\Leftrightarrow \{e\}(n)\downarrow$$

- We will show that Halt is undecidable.

Example

- Let $e = \text{encode}(T)$, where T is the Turing machine T

$$I_0 = \text{ifzero}(0, 0)$$

- If input is > 0 , the program terminates immediately, and R_0 remains unchanged, so

$$\{e\}(k) \simeq T^{(1)}(k) \simeq k$$

for $k > 0$.

- If input is $= 0$, the program loops for ever.
Therefore

$$\{e\}(0) \simeq T^{(1)}(0) \uparrow .$$

- Therefore $\text{Halt}(e, k)$ holds for $k > 0$ and does not hold for $k = 0$.

Remark

- Below we will see: Halt is undecidable.
- However, the following function WeakHalt is computable:

$$\text{WeakHalt}(e, n) \simeq \begin{cases} 1 & \text{if } \{e\}(n) \downarrow \\ \perp & \text{otherwise} \end{cases}$$

- Computed as follows:
 - First check whether $e = \text{encode}(T)$ for some Turing machine T .
 - If not, enter an infinite loop.
 - Otherwise, simulate T with input n .
 - If simulation stops, output 1, otherwise the program loops for ever.

Question

- What is $\text{WeakHalt}(e, n)$, where e is a code for the Turing machine

$$I_0 = \text{ifzero}(0, 0) \text{ ?}$$

Theorem 4.6

Theorem 4.6 *The halting problem is undecidable.*

Proof:

- **Assume** the Halting problem is decidable
i.e. assume that we can decide using a Turing machine whether $\{e\}(n) \downarrow$ holds.
- We will define below a computable function $f : \mathbb{N} \xrightarrow{\sim} \mathbb{N}$,
s.t. $f \neq \{e\}$.
- Therefore f cannot be computed by the Turing machine with code e for any e , i.e. f is noncomputable.
- Therefore we obtain a **contradiction**.

Proof of Theorem 4.6

- We argue similarly as in the proof of $\mathbb{N} \not\cong \mathcal{P}(\mathbb{N})$
 - We define $f(e)$ in such a way that $f = \{e\}$ is violated by having $f(e) \neq \{e\}(e)$.
- If $\{e\}(e) \downarrow$, then we let $f(e) \uparrow$.
- If $\{e\}(e) \uparrow$, we let $f(e) \downarrow$, e.g. by defining $f(e) \simeq 0$ (any other defined result would be appropriate as well).
- So we define

$$f(e) \simeq \begin{cases} \perp, & \text{if } \{e\}(e) \downarrow \\ 0, & \text{if } \{e\}(e) \uparrow \end{cases}$$

Proof of Theorem 4.6

$$f(e) \simeq \begin{cases} \perp, & \text{if } \{e\}(e) \downarrow \\ 0, & \text{if } \{e\}(e) \uparrow \end{cases}$$

- We obtain $f(e) \downarrow \Leftrightarrow \{e\}(e) \uparrow$. (*)
- Since Halt is decidable, f is computable (Exercise: show that f is computable by a Turing machine, assuming a Turing machine for Halt).
- Therefore $f = \{e\}$ for some e .
- But then by (*)

$$f(e) \downarrow \stackrel{(*)}{\Leftrightarrow} \{e\}(e) \uparrow \stackrel{f=\{e\}}{\Leftrightarrow} f(e) \uparrow$$

a contradiction.

Proof of Theorem 4.6

The complete proof on one slide is as follows:

- Assume Halt were decidable.
- Define

$$f(e) \simeq \begin{cases} \perp, & \text{if } \{e\}(e) \downarrow \\ 0, & \text{if } \{e\}(e) \uparrow \end{cases}$$

- By Halt decidable, we obtain f is computable, so $f = \{e\}$ for some e .
- But then

$$f(e) \downarrow \stackrel{\text{Def of } f}{\Leftrightarrow} \{e\}(e) \uparrow \stackrel{f=\{e\}}{\Leftrightarrow} f(e) \uparrow$$

Remark

- The above proof can easily be adapted to any reasonable programming language, in which one can define all intuitively computable functions.
- Such programming languages are called Turing-complete languages.
 - Babbage's machine was, if one removes the restriction to finite memory, Turing-complete, since it had a conditional jump.
- For standard Turing complete languages, the unsolvability of the Turing-halting problem means: it is not possible to write a program, which checks, whether a program on given input terminates.

Halting Problem with no Inputs

- **Theorem 4.7:** It is undecidable, whether a Turing machine started with a blank tape terminates.

- **Proof:**

- Let

$\text{Halt}'(e) :\Leftrightarrow e$ is a code for a Turing machine T
and T started with a blank tape
terminates

- Assume Halt' were decidable.

Halting Problem with no Inputs

- Then we can decide $\text{Halt}(e, n)$ as follows:
 - Assume inputs e, n .
 - If e is not a code for a Turing machine, we return 0.
 - Otherwise, let $\text{encode}(T) = e$.
 - Define a Turing machine V as follows:
 - V first writes $\text{bin}(n)$ on the tape and moves head to the left most bit of $\text{bin}(n)$.
 - Then it executes the Turing machine T .
 - We have
 - V , run with blank tape, terminates
 - iff T run with tape containing $\text{bin}(n)$ terminates
 - iff $T^{(1)}(n) \downarrow$
 - iff $\{e\}(n) \downarrow$.

Halting Problem with no Inputs

V , run with blank tape, terminates iff $\{e\}(n)\downarrow$.

- Let $\text{encode}(V) = e'$. Then

$$\text{Halt}'(e') \Leftrightarrow \text{Halt}(e, n)$$

- Therefore using the decidability of Halt' we can decide $\text{Halt}(e, n)$.
- So we have decided Halt , a contradiction.