CS_226
COMPUTABILITY THEORY
Exam May/June 2008

(*Attempt* **2** *questions out of 3*)

Question 1. *(Countable Sets)*

(a) What does it mean for a set to be countable? Is every finite set countable? Is every infinite set countable? Justify your answers.

[8 marks]

(b) Which of the following sets are countable?

(i) The set of rational numbers, which are not integers, i.e. $\mathbb{Q} \setminus \mathbb{Z}$.

(ii) The set of irrational numbers, i.e. $\mathbb{R} \setminus \mathbb{Q}$.

(iii) The set of functions $f : \{0, 1\} \to \{0, 1, 2\}$.

(iv) The set of functions $f : \mathbb{N} \to \{0, 1, 2\}$.

(v) The set of functions $f : \mathbb{N} \to \{0\}$.

Justify your answers.

[10 marks]

(c) In the lecture the following statement has been shown:

Theorem A set B is countable if and only if there is an injective function $f : B \to \mathbb{N}$.

Show directly using this theorem, that if B is countable and $g : A \to B$ is injective, then A is countable as well.

[7 marks]

Please turn over for Question 2.

Question 2. *(URMs and Turing Machines)*

(a) Define the notion of a URM. Your description should contain a description of the URM instructions and how a URM is executed. If U is a URM, it defines a function $\mathrm{U}^{(k)} : \mathbb{N}^k \xrightarrow{\sim} \mathbb{N}$ for every $k \in \mathbb{N}$. Explain, how $\mathrm{U}^{(k)}(a_0, \ldots, a_{k-1})$ is computed.

[9 marks]

(b) Consider the following URM program U:

$$\begin{array}{lcl} \mathrm{I}_0 & = & \mathrm{ifzero}(1,5) \\ \mathrm{I}_1 & = & \mathrm{succ}(0) \\ \mathrm{I}_2 & = & \mathrm{succ}(0) \\ \mathrm{I}_3 & = & \mathrm{pred}(1) \\ \mathrm{I}_4 & = & \mathrm{ifzero}(2,0) \end{array}$$

Determine the function $\mathrm{U}^{(2)} : \mathbb{N}^2 \xrightarrow{\sim} \mathbb{N}$ computed by U. Justify your answer. One way of answering this question is by systematically transforming this program into a higher level programming language first, and then to determine what the resulting program does.

[8 marks]

(c) Introduce a **Turing machine** T which computes the function $f : \mathbb{N} \xrightarrow{\sim} \mathbb{N}$, defined by

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

Explain, why your solution computes f.

[8 marks]

Please turn over for Question 3.

Question 3. *(Primitive Recursive and Recursive Functions and Relations)*

(a) Define the set of primitive recursive functions. Give an example of a function which is recursive but not primitive recursive.

[7 marks]

(b) Show, using the definition of primitive recursive functions directly, that the function

$$f : \mathbb{N}^2 \to \mathbb{N} \ , \quad f(x, y) := 2x + y$$

is primitive recursive.

[6 marks]

(c) (i) Define the notion of a primitive recursive relation.

(ii) Let P, Q be primitive recursive relations. Show that the intersection of P and Q, i.e. the relation R defined by $R(\vec{x}) :\Leftrightarrow P(\vec{x}) \cap Q(\vec{x})$ is primitive recursive as well.

(iii) Determine, whether the following statement (which expresses that the primitive recursive relations are closed under unbounded existential quantification) is true of false: If an $n+1$-ary relation P is primitive recursive, so is the n-ary relation Q defined by $Q(\vec{x}) \Leftrightarrow \exists y.P(\vec{x}, y)$. Justify your answer.

[12 marks]