CS_226
COMPUTABILITY THEORY
Exam May/June 2007
(*Attempt* **2** *questions out of 3*)

Question 1. *(Cardinality, Countable Sets, and Diagonalisation Arguments)*

(a) What does it mean for a set to be countable? Give an example of an infinite set which is countable, and of an infinite set which is uncountable.

[4 marks]

(b) Define what it means for a function $f : A \to B$ to be injective. Prove that, if $f : A \to B$ and $g : B \to C$ are injective, so is $g \circ f : A \to C$.

[4 marks]

(c) If $f : A \to B$ is injective, there is a relationship between the properties "A is countable" and "B is countable". Which one?

[3 marks]

(d) Which of the following sets is countable? Justify your answer:

(i) $\mathbb{Q}$.

(ii) $\mathbb{R} \setminus \mathbb{Q}$.

(iii) $\{x \in \mathbb{R} \mid x^2 = 1\}$.

(iv) $\{f : \mathbb{N} \to \mathbb{N} \mid f \text{ computable}\}$.

[8 marks]

(e) Assume a computable function $h : \mathbb{N}^2 \to \mathbb{N}$. Define for $e \in \mathbb{N}$ the function $\{e\}'' : \mathbb{N} \to \mathbb{N}$, $\{e\}''(n) := h(e, n)$. Show that there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $f \neq \{e\}''$ for all $e \in \mathbb{N}$.

[6 marks]

Please turn over for Question 2.

Question 2. *(URMs)*

(a) Define the notion of a URM. Your definition should include the instructions of the URM, the architecture of a URM, and how a URM program is executed. Explain how to compute the function $\mathrm{U}^{(1)} : \mathbb{N} \stackrel{\sim}{\rightarrow} \mathbb{N}$, if U is a URM.

[6 marks]

(b) Consider the following URM program U:

$$\begin{array}{lcl} \mathrm{I}_0 & = & \mathrm{ifzero}(0, 5) \\ \mathrm{I}_1 & = & \mathrm{ifzero}(0, 4) \\ \mathrm{I}_2 & = & \mathrm{pred}(0) \\ \mathrm{I}_3 & = & \mathrm{ifzero}(1, 1) \\ \mathrm{I}_4 & = & \mathrm{succ}(0) \end{array}$$

Determine the function $\mathrm{U}^{(1)} : \mathbb{N} \stackrel{\sim}{\rightarrow} \mathbb{N}$ computed by U. Justify your answer.

[8 marks]

(c) Let $n, aux \in \mathbb{N}$. Define a URM program U, which when executed with initially having register R_{aux} containing 0, terminates with all registers having their original value except for register R_n, which is set to 0. So U essentially sets register R_n to 0. Explain, why your program has the desired behaviour.

[3 marks]

(d) Assume an extension of the language of the URM by one additional instruction zero(n). zero(n) is executed by setting register R_n to 0 and then continuing executing the next instruction. For a program U′ in the extended language we can define $\mathrm{U}'^{(1)}$ as for URM-programs. Show how to translate a program U′ of the extended language into a URM program U of the original language of URMs in such a way that $\mathrm{U}'^{(1)} = \mathrm{U}^{(1)}$. Explain your answer and why $\mathrm{U}^{(1)} = \mathrm{U}'^{(1)}$. A formal proof is not required. You can assume that there exists a solution for Question 2 (c).

[8 marks]

Please turn over for Question 3.

Question 3. *(Primitive Recursive Functions and Equivalence Theorem)*

(a) Define the notion of a primitive recursive function. Define a total computable function, which is not primitive recursive.

[6 marks]

(b) Let $f : \mathbb{N}^2 \to \mathbb{N}$, $f(n, m) := 3 \cdot m + n$. Show that this function is primitive recursive by using directly the principle of primitive recursion, without making use of any lemmas shown in the module.

[4 marks]

(c) Let $g : \mathbb{N} \to \mathbb{N}$, $g(0) := g(1) := g(2) := 1$, $g(n) := g(n-3) + g(n-2) + g(n-1)$ for $n \geq 3$. Show that g is primitive recursive. You can make use of any lemmas given in the lecture.

[4 marks]

(d) Which of the following statements is true? Justify your answers:

(i) If $A, B \subseteq \mathbb{N}^n$ are primitive recursive, so is $A \cup B$.

(ii) If $A \subseteq \mathbb{N}^{n+1}$ is primitive recursive, so is $\{(\vec{x}, y) \in \mathbb{N}^{n+1} \mid \exists z < y.(\vec{x}, z) \in A\}$.

(iii) If $A \subseteq \mathbb{N}^{n+1}$ is primitive recursive, so is $\{\vec{x} \in \mathbb{N}^n \mid \exists z.(\vec{x}, z) \in A\}$.

[6 marks]

(e) Give an outline of a proof that every primitive recursive function is URM computable. You only need to state for which functions one needs to show that they are URM computable, without actually proving those statements.

Hint: You have seen a proof that every partial recursive function is URM computable. Use the fact that the primitive recursive functions form a subset of the partial recursive functions.

[5 marks]